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# Application of Monte Carlo Methods in Finance

Lyon, October 2002

# Overview

- Prerequisites
  - the time value of money
  - stock prices as stochastic processes
- Financial Derivatives
  - definition and examples
  - valuation
- Application of Monte Carlo methods for valuing options
  - European style options
  - improving convergence
  - Asian style and look-back options
- Value at Risk (VaR)
  - definition and examples
  - using Monte Carlo techniques to calculate the Value-at-Risk

# Assumptions used throughout the course

- Investors do not default.
  - Hence, if we borrow money to an investor or enter into a financial contract with him, we can be sure that he is able to meet his obligations.
- Investors can issue any kind of security at zero transaction costs.
  - In the real world, if you buy or sell securities you have to pay money to the intermediaries (broker, exchange).
  - However, for large financial institutions these costs are much lower than for private investors.
- Money can be invested or borrowed at the same interest rate.
  - Again, this assumption holds best for financial institutions, while for private investors the rate of borrowed usually exceeds the rate of investing.

# The time value of money

Sophisticated investors can use a whole range of financial instruments, from stocks to options. The most simple of them is the **bank or money market account**.
For any money invested in a bank account the investor earns interest on the money.

$$t = 0 \qquad N(0) = N$$

$$t = 1y \qquad N(1y) = (1 + r(1y))N = N + r(1y)N$$

$$interest = N\,r(1y)$$

Hence, if you have an euro, you can either use it for today's consumption, or put it in your bank account and earn interest on it. If you do the later, you will have more money for consumption available in the future.

# Remarks

- Invested money is not available for consumption at t=0.
  - earned interest = compensation for deferred consumption
- In the real world the interest rate r depends on the length of the investment period (how long your money is tied in your account)
  - the longer the period an investor gives up control over his or her money, the higher the interest rate will be she or he expects
  - however, in the real world, interest rates depend on a number of factors
    - short term interest rates are set by the central banks
    - long term interest rates depend on the inflation expectations of the market
    - the credit rating of the party issuing a financial instrument, for instance a fixed coupon bond, also plays a very important role (Argentinia does not pay the same rate as France)

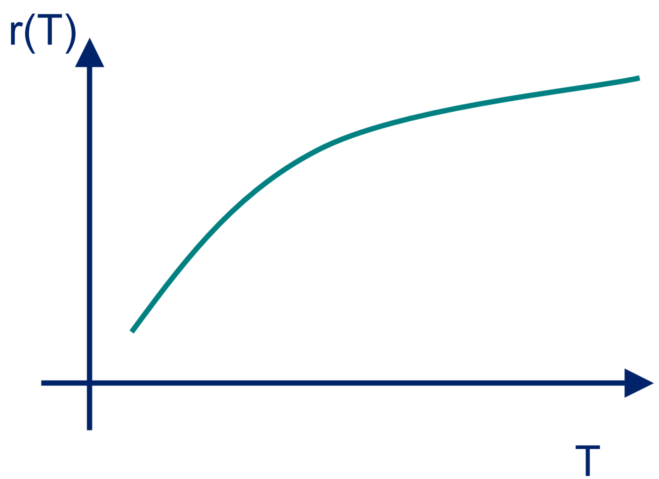
  

# generalization to investments of any maturity

$$N(T) = (1 + r(T))^T N(0)$$

$$e.g.: N(2y) = (1 + r(2y))(1 + r(2y))N(0)$$

Note that the interest rate depends on the maturity of the investment.




typical interest rate curve

# interest bearing financial instruments

- fixed rate or fixed coupon bonds
  - security issued by a country, bank or corporate paying a fixed coupon on a regular basis (usually annual or semi-annual)
- floating rate note
  - like a fixed rate bond, however, here the interest rate for any investment period is fixed at the beginning of the period
- money market instruments
  - usually interest rate instruments with a short maturity (<= 3 months) where all interest is paid together with the principal at maturity
- interest rate swaps
  - instruments which allow investors to exchange fixed rate against floating rate payments

# discount factors

Assume that investor A holds a security entitling him to receive 1 Euro at t=T. He wants to sell the security to a second investor B. How much is the security worth?

Assume that the security is worth x Euro. Then investor B has two options available:

1. Buy the security from investor A and receive 1 Euro at t=T.

2. Invest x Euro into a bank account and receive $z = (1 + r(T))^T x$ Euro at t=T.

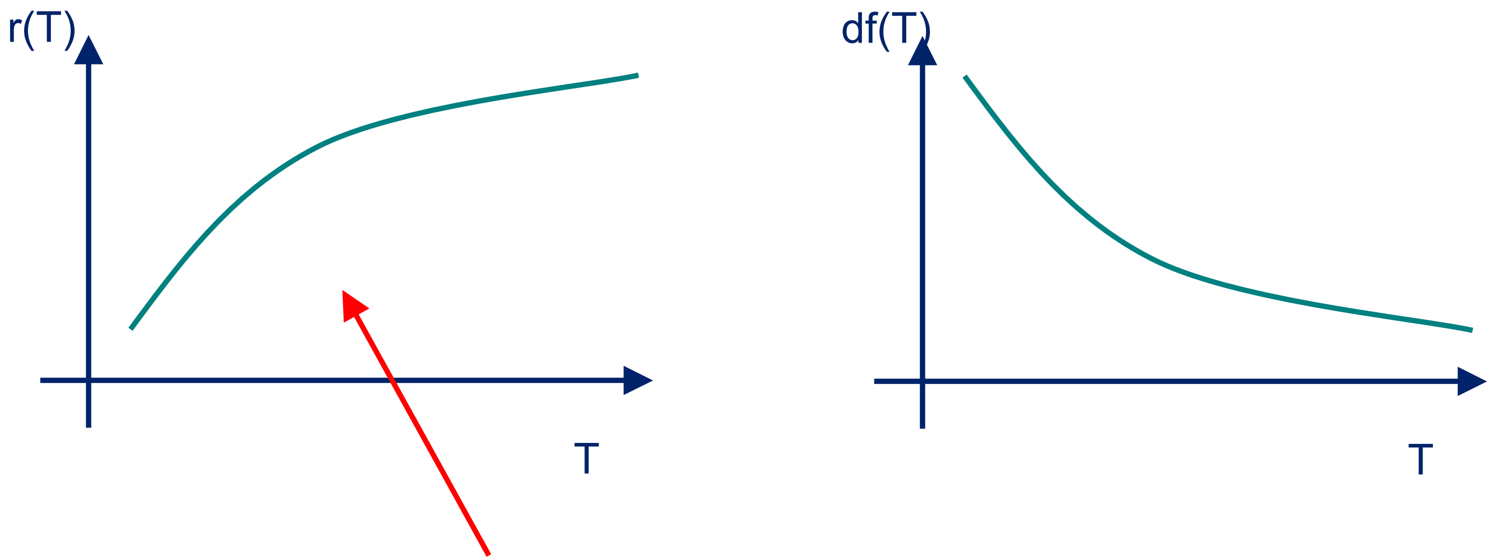
Clearly, if no arbitrage opportunity exists and the security trades at a fair price, z should equal 1.

$$\Rightarrow z = 1$$

$$\Rightarrow x = \frac{1}{(1 + r(T))^T} = df(T)$$

discount factor = today's value of one Euro paid at t=T

# discount factors vs. interest rate curves




But look out, other shapes exist in the real world!

# compounding methods

In the markets a huge variety of different compounding methods exist (usually annoying the quantitative analyst).

**annual**

$$df(T) = \frac{1}{(1+r(T))^T}$$

**simple**

$$df(T) = \frac{1}{1+r(T)T}$$

**linear**

$$df(T) = 1 - r(T)T$$

**semi annual**

$$df(T) = \frac{1}{\left(1+\frac{r(T)}{2}\right)^{2T}}$$

**quarter annual**

$$df(T) = \frac{1}{\left(1+\frac{r(T)}{4}\right)^{4T}}$$

**continuous**

$$df(T) = e^{-r(T)T}$$

# Remarks

- While the interest rate for a given date depend on the compounding method the discount factor, which measures the worth of one Euro received at this date, must be same for all compounding methods.
- Besides different compounding methods the financial community has also invented a number of methods for calculating the time in years between two dates. These two factors can make calculations rather complicated.
- From now on, we will use the continuous compounding method, because of mathematical tractability. We will also assume that the interest rate is the same for all maturities, hence we can drop the time dependence of r.

# Remarks (cont.)

- In practice the rate for lending and investing is not the same. In particular the rate paid by an investor when borrowing money depends on her or his credit worthiness. The interest rate will increase with the probability of default.
- Interest rates also depend on the currency. Since exchange rate fluctuate different interest rate in different countries do not allow for riskfree investment strategies, which would yield returns above the current market rate.

# stocks in a nutshell

- Stocks usually represent a share in the equity of a company.
  - thus the shareholders own and control the company
  - part of the earnings are distributed to the shareholders
  - however, over kinds of shares exist, e.g. preferred shares (no control, but higher dividend)
- A stock index is a weighted average of share prices.
  - usually consists of the largest public companies of a regional market (e.g. Spain) or industry segment (European Telecom Companies)
    - France: CAC40
    - Germany: DAX
    - USA: Dow Jones
    - …
  - note: different kinds of indices exist, depending on whether dividends are included (=performance indices) or not
- Stocks are traded on exchanges.
  - for liquid stocks at any given point in time a whole range of bid and ask offers exist, assuring that investors can buy and sell these securities

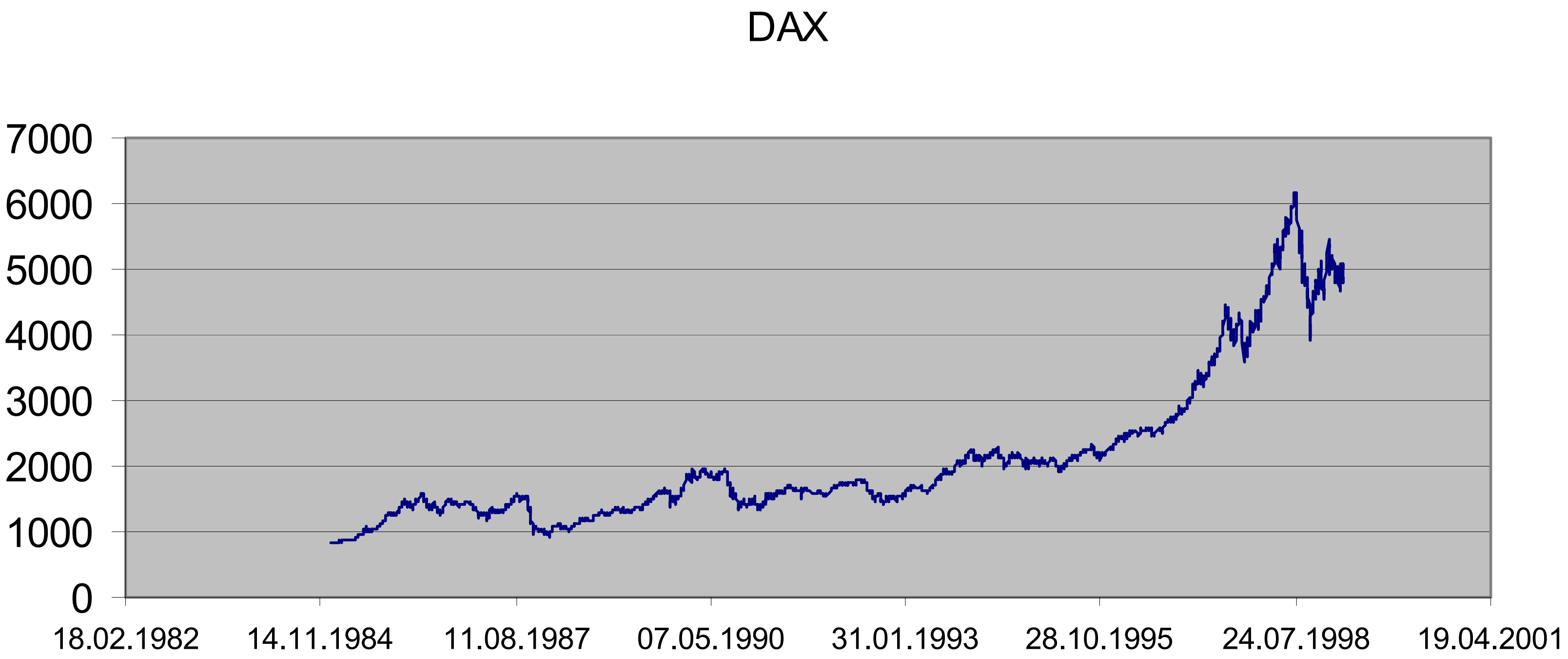
# stock prices as stochastic processes

In financial mathematics stock prices are assumed to follow a Markov process, which is consistent with the so called weak form of market efficiency.

- The current price of a stock or stock index should reflect all available information. If this was not the case, then somebody would be able to make a profit by either buying or short selling the stock or the index. This in turn would push the price higher or lower, resulting in a new, efficient price.
- New information, arriving at random, will result in random moves of the stock price.

**So if stock prices follow a stochastic process, which model should we use?**

# price history of the DAX (German stock index)



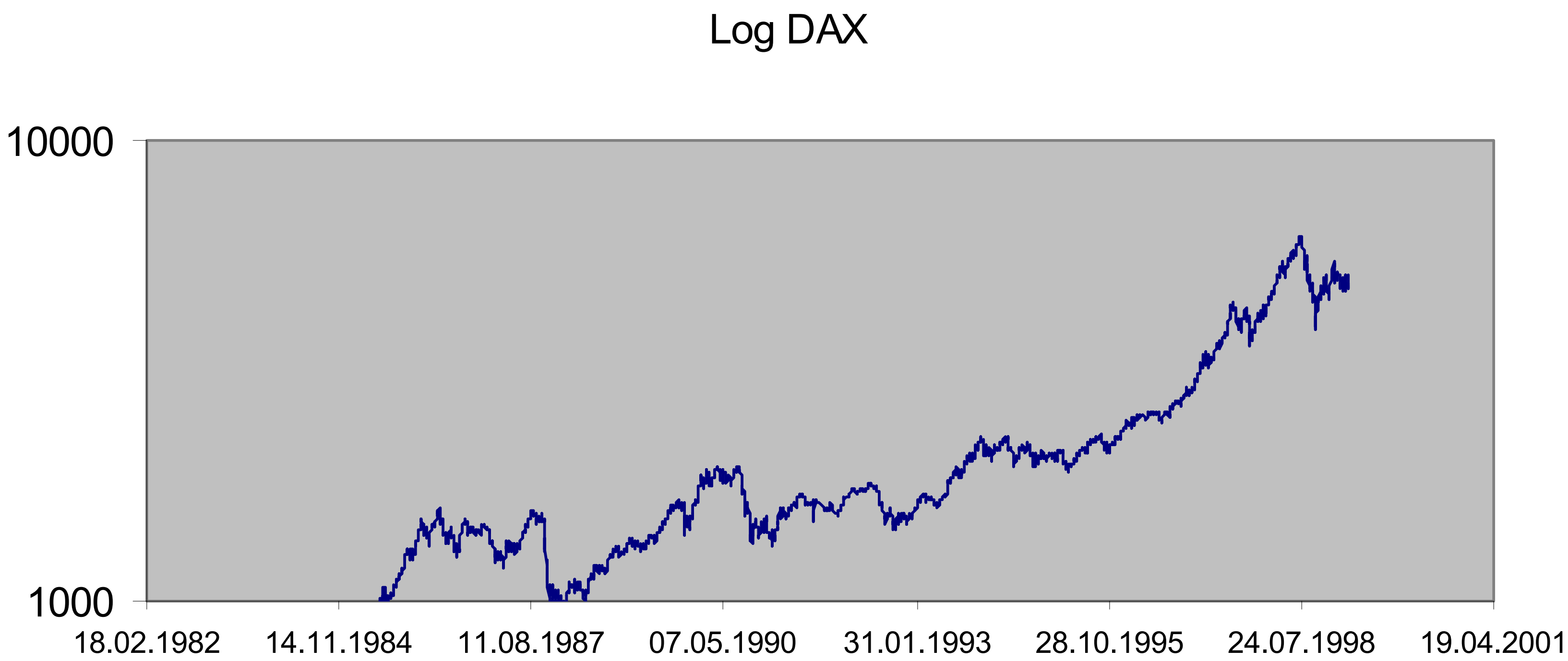
# Remarks

- There is a clear upward trend, resulting on average in a long time growth.
- The trend is non-linear.

Would we expect share prices to grow linear?

No, because then the relative growth rate would depend on the share price.

# growth of the DAX on a logarithmic scale




On a logarithmic scale a linear trend emerges, hence we model $\frac{\Delta S}{S}$

# Review of the Wiener process

A variable x follows a Wiener process if:

1. The change $\Delta z$ over a small period of time, $\Delta t$ , is given by:

   $$\Delta z = \varepsilon\sqrt{\Delta t}$$

   where $\varepsilon$ is a random drawing from a standardized normal distribution.

2. For any two time intervals the changes, $\Delta z_i$ and , $\Delta z_j$ are independent.

# Review of the Wiener process (cont.)

Hence:

$$E(\Delta z) = 0$$
$$E(\Delta z^2) = \Delta t$$
$$E(\Delta z_i \Delta z_j) = 0, \ if \ i \neq j$$

Taking the limit: $\Delta t \to 0$

$$dz = \varepsilon \sqrt{dt}$$

Where the following relationships hold:

$$E(dz) = 0$$
$$E(dz^2) = dt$$

Furthermore:

$$Z_t = \int_0^t dz_t \sim N\left(0, \sqrt{t}\right)$$

$$\int_0^T f(t) dz_t \sim N\left(0, \sqrt{\int_0^T f(t)^2 \, dt}\right)$$

# generalized Wiener process

- Adding a drift to the Wiener process and allowing arbitrary volatilities we arrive at the generalized Wiener process, which is given by:

$$dx = a\,dt + b\,dz$$

- Where a and b are constants, and z follows a Wiener process.

# Ito Process

- If a and b are not constants but depend on the underlying variable, we call the process an Ito process.

$$dx = a(x,t)dt + b(x,t)dz$$

- In this case:

$$E(dx) = a(x,t)dt$$
$$E(dx^2) = b(x,t)^2 dt$$

- Hence $dx^2$ is of order dt.

# Ito's Lemma

- If x follows an Ito process and f is a function of x and t (f=f(x,t)). What process describes the evolution of f (hence, can we calculate df)?

$$df = f(x+dx, t+dt) - f(x,t)$$

$$= \frac{\partial f}{\partial x} dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dx^2 + \frac{\partial f}{\partial t} dt$$

- Since dx is of order $dt^{0.5}$, $dx^2$ must be of order dt. Hence the expansion must include the term of order $dx^2$. In order to simplify the above expression we need the following relationship:

$$dx^2 = b(x,t)^2\, dt$$

exact proof is behind the scope of the lecture, but remember that Ito calculus is just a short hand form of the underlying mathematics

# Ito's Lemma (cont.)

- Using the last result, we obtain:

$$df = \frac{\partial f}{\partial x} dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dx^2 + \frac{\partial f}{\partial t} dt$$

$$= \left( \frac{\partial f}{\partial x} a(x,t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} b(x,t)^2 + \frac{\partial f}{\partial t} \right) dt + \frac{\partial f}{\partial x} b(x,t) dz$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} b(x,t)^2 dt$$

- This formula is called Ito's Lemma.

# Basic model of financial literature

- As we saw above, stock prices exhibit a logarithmic trend, hence it makes more sense to model the percentage growth of the stock price. The most simple model, one can write down then is:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz \Leftrightarrow dS = \mu\, S\, dt + \sigma\, S\, dz$$

- Where: $\mu$ = growth rate of the stock

  $\sigma\, S$ = price volatility of the stock

- And $\mu$ and $\sigma$ are constants.

# Reformulation of the model

- The model is the basic model of the financial literature. Here S follows an Ito process. However, it is often more convenient to work with the logarithm of S, ln(S). A formula for the logarithm can be derived by applying Ito's lemma:

$$d \ln S = \left( \frac{\partial \ln S}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 \ln S}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial \ln S}{\partial S} \sigma S \, dz$$

$$= \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma \, dz$$

- Hence the logarithm follows a generalized Wiener process.

# Working in finite time

- One usually has to work in finite time Δt. Hence we need a formula for ΔlnS.

$$\ln S(t+\Delta t) = \ln S(t) + \Delta \ln S \Leftrightarrow S(t+\Delta t) = S(t)\exp(\Delta \ln S)$$

$$\Delta \ln S = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\,\Delta z$$

$$\Rightarrow \ln S(t+\Delta t) = \left(\ln S(t) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t\right) + \sigma\,\Delta z$$

$$\Rightarrow S(t+\Delta t) = S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\,\Delta z\right)$$

# Remarks

- One should always model lnS (if possible), because it follows a Wiener process where drift and volatility are constants. Hence, dlnS follows a classic random walk (with drift), and one does not run into numerical inaccuracies when switching to finite times. This is not the case for an Ito process, here neither drift nor volatility are constants.
- In particular when simulating the behavior of the stock one should use dlnS and refrain from using $\frac{\Delta S}{S}$. The latter will invariably lead to numerical errors. For instance, S can become negative.

# Integrating dlnS

- One can also integrate dlnS:

$$\ln S(T) = \ln S(t) + \left( \mu - \frac{1}{2}\sigma^2 \right)(T - t) + \sigma \int_t^T dz_t$$

$$S(T) = S(t) \exp\left( \left( \mu - \frac{1}{2}\sigma^2 \right)(T - t) + \sigma \int_t^T dz_t \right)$$

- The above formula is useful, when one is only interested in the value of S at t=T. Remember that:

$$Z_T = \int_t^T dz_t \sim N\left(0, \sqrt{T - t}\right)$$

**So what do we have to do, when we want to simulate S(T)?**

# Expected Value of S

- Further insight into the necessity of the drift correction can be gained by calculating the expected value of S at t=T:

$$E(S(T)) = S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)\frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{+\infty} dZ\,\exp(\sigma Z)\exp\left(-\frac{Z^2}{2(T-t)}\right)$$

$$= S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)\frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{+\infty} dZ\,\exp\left(-\frac{Z^2 - 2\sigma(T-t)}{2(T-t)}\right)$$

$$= S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)\frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{+\infty} dZ\,\exp\left(-\frac{(Z-\sigma(T-t))^2 - \sigma^2(T-t)^2}{2(T-t)}\right)$$

$$= S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)\exp\left(\frac{1}{2}\sigma^2(T-t)\right)\frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{+\infty} dZ\;\exp\left(-\frac{Z^2}{2(T-t)}\right)$$

$$= S(t)\exp(\mu(T-t))$$

# Remarks

- Without the drift correction we would not get the correct result, because:

$$E(\exp(Z)) \neq \exp(E(Z))$$

- When applying the above model, the growth rate poses the greatest problem. It is very difficult to measure, because one needs long time intervals to dampen out the effects of the stock's volatility. But as can be seen from the picture above, the growth rate seems to vary over long intervals of time. This is consistent with the observation that the growth rate of the economy is not constant over time either.

Solution? Eliminate the growth rate.

# Derivatives

*Definition*:

- A derivative is a financial instrument whose value depends on the value of a underlying variable.

- The underlying variable can be the price of a stock, but also an interest or an exchange rate. Also more complex situations exist, where several financial instruments serve as underlying or the derivative depends on environmental factors, like the average temperature, or even economic conditions, like the tax rate or inflation.

- Here we will only consider derivatives depending on stock or stock index prices. To further simplify things, we will ignore dividends.

# Forward Contracts

- One of the simplest derivative is the forward contract. Here one investor agrees to purchase a share from a second investor at a time T in the future at a fixed price K, the forward price.


at t=T A delivers the share to B, who has to pay the price fixed at t=0 in return

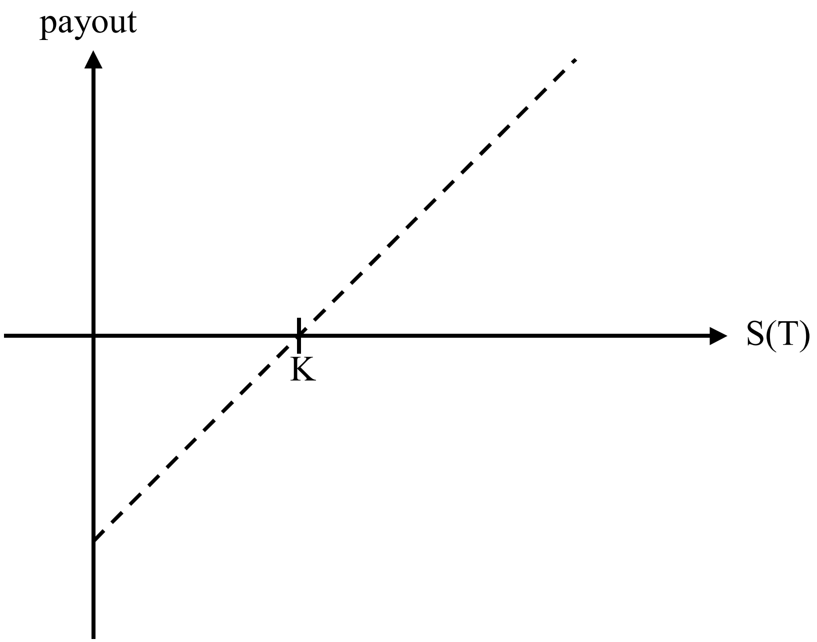
at t=0 B agrees to purchase from A a share at t=T at a price K.

# Remarks

- The investor buying the stock at T is said to be long the forward contract, while the investor selling the stock at T is short the forward contract.
- No money is exchanged at the beginning, hence it does cost nothing to enter the forward contract.
  - However, you are obliged to deliver the security or buy it at t=T respectively.
- Sometimes the share is not really exchanged at T (physical settlement), instead one investor pays the other the difference between the current stock price and the agreed price K.
  - If the current stock price at T is above K, the investor long the contract receives money from the investor short the forward contract
  - Otherwise the investor long the contract has to pay money to the investor short the contract.

# Payout-Profile

The following diagram shows the payout profile of a long position in the forward contract at t=T depending on the stock price at t=T.



# The forward price

- Note, since it costs nothing to enter the forward contract at t=T, the agreed price K, which is also called the forward price, must allow no arbitrage.
- Here arbitrage means, that it is possible to construct a portfolio, earning for certain a rate above the risk free rate. Hence, K must be chosen so that such a portfolio cannot be constructed.

# Constructing the forward price

Assume that an investor has S(0) Euro to invest. He needs the money at t=T, and therefore is not willing to take any risks. He basically has two options:

- Invest the money in a bank account (portfolio A).
- Buy a stock, but at the same time enter a forward contract, to sell the stock at t=T (portfolio B).

| | investment at t=0 | payoff at t=T |
|---|---|---|
| **portfolio A** | S(0) in bank account | S(0)exp(r*T) |
| **portfolio B** | long one share<br>short one forward contract | K |

# Constructing the forward price (cont.)

Since both portfolios are deterministic, and have initial set up costs, they should earn the same amount of money. Hence:

$$K = S(0)\exp(rT)$$

Suppose that this was not the case, and K would actually exceed the arbitrage free price. In this case you could do the following:

- borrow S(0) Euro from the bank (implying that you have to pay back S(0)exp(rT) at t=T)
- buy the stock (which costs you S(0) Euro)
- enter the forward contract

Note: Your initial set up costs are zero.

At t=T you would:

- deliver the stock and receive K Euro
- use the proceeds to pay back your debt

Since K exceeds your debt, you make a certain profit

# Options

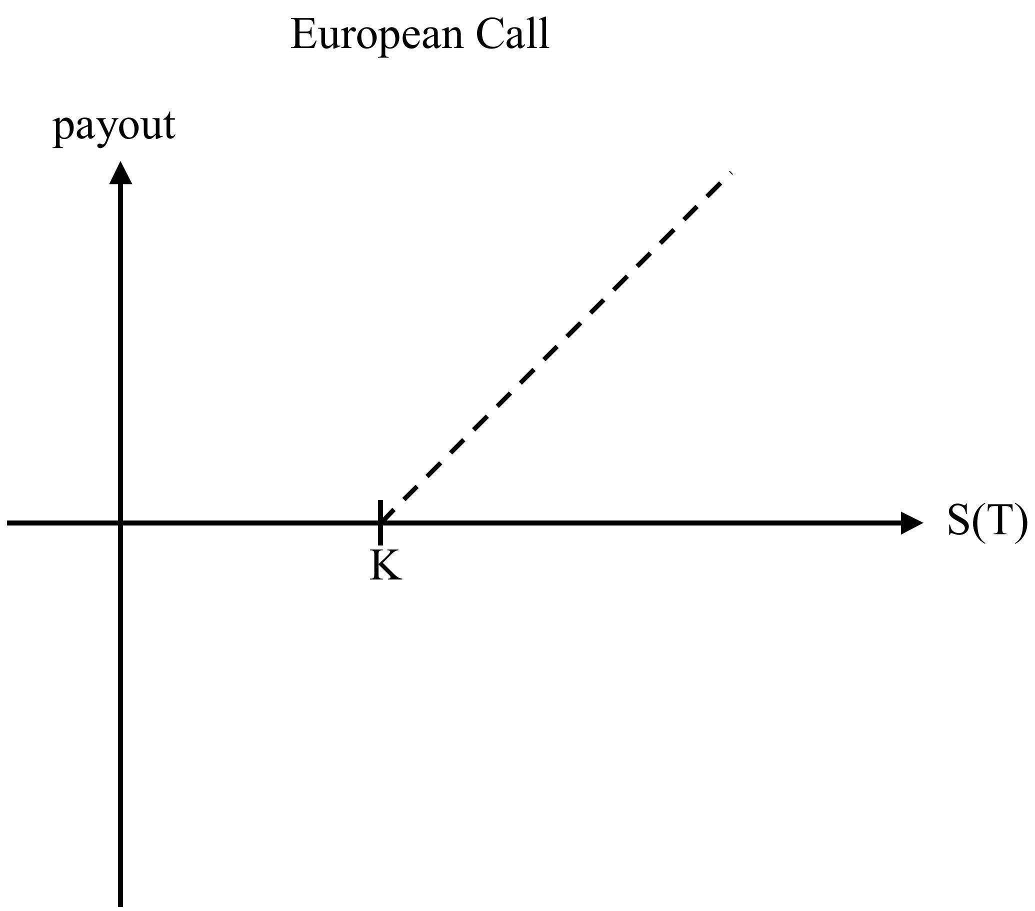
- Basic features:
  - only one side decides whether contract is exercised
    - however, more complicated situations exist
  - options cost money
- While forward contracts must be executed by both investors at t=T, options give one investor, the investor long the particular option, the option to execute the contract at t=T (or at an earlier point in time, if the option is American or Bermudan style). Hence, the investor short the contract has no influence on whether the contract is executed at maturity. Obviously the investor long the contract will only execute the option if it is in the money.

# European Call

- A European call on a stock gives the investor the right to buy the stock at maturity (t=T) at a predetermined price K, the strike price (physical settlement).
- Under what circumstances will the investor execute the option at maturity?
  - Answer: If S(T)>K
- Besides physical settlement, where the stock is exchanged between the investors, also cash settlement exists, where in the event of execution the short investor pays to the long investor the difference between the current stock price and the strike price. In this case the payout profile is given by:

$$payout = \max(0, S(T) - K)$$

# Payout profile of European Call



# Option Styles and the right to Exercise

The "style" of an option depends on when an investor is allowed to exercise it:

| | |
|---|---|
| European style | Can only be exercised at maturity ($t_{excercise}$=T) |
| American style | Can be exercised at any point in time during the liftetime of the option ($0 \leq t_{exercise} \leq T$) |
| Bermudan style | A Bermudan style option can only be exercised at certain points in time (e.g. $t \in [T_1, T_2, T_3, T_4]$, where $0 \leq T_i \leq T$) |

However, for reasons of simplicity we will only consider European style options.

Note, that in particular American style options are not very well suited for Monte Carlo methods.

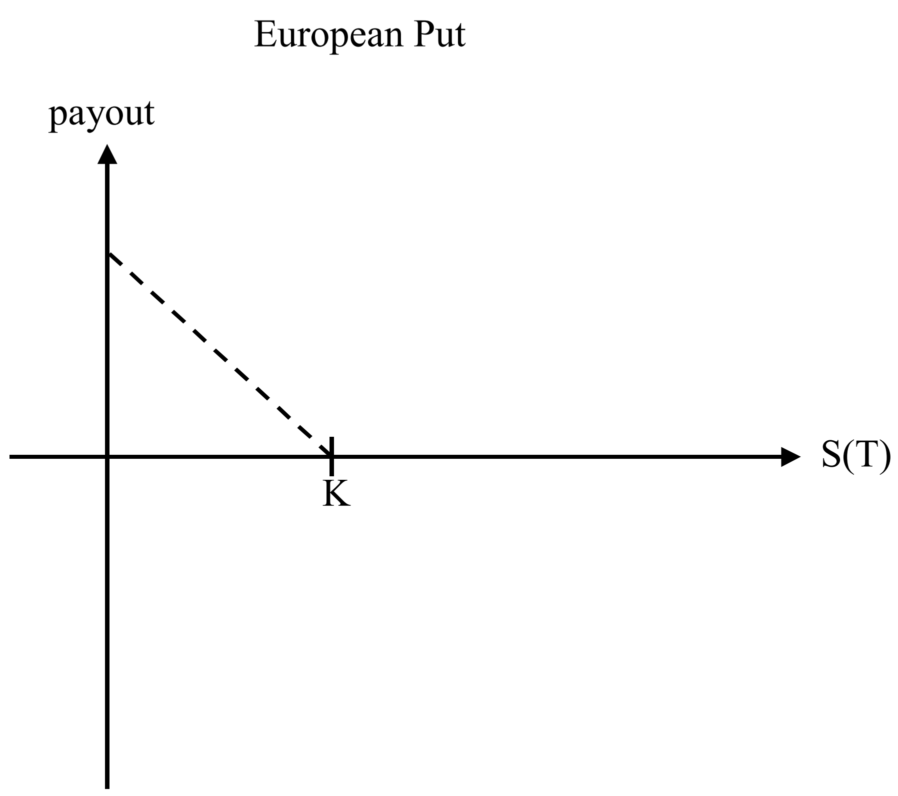
# European Put

- Here, in the case of cash settlement, the investor receives the positive difference between the strike price and the stock price, if the stock price is below the strike price. Hence, the payout profile is given by:

$$payout = \max(0, K - S(T))$$

- While a call option is a bet on rising stock prices, a put option is a bet on falling stock prices.

# Payout profile of European Put



# Valuation of Derivatives

- Advanced derivatives like options are not as easy to evaluate as for instance forward contracts.
  - Fortunately one is required to apply some advanced mathematics in order to derive an analytical formula.
  - However, for most options no analytical formula exists. Hence, it is necessary to use numerical methods.

- The starting point of the valuation of forward contracts is the following portfolio:
  - short one option
  - long Δ stocks
- Hence the value of your portfolio is given by: $V = -c + \Delta S$

# Derivation of Black-Scholes-Merton equation

- We now look at the changes in portfolio over an infinitesimal time step dt.

$$dV = -dc + \Delta dS$$

$$= -\left( \frac{\partial c}{\partial S} dS + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 dt + \frac{\partial c}{\partial t} dt \right) + \Delta dS$$

- Where in the last step Ito's Lemma was applied to calculate dc.
- We now chose Δ so that the portfolio becomes risk free, which means that dS is eliminated from the above equation. This requirement can only be met by choosing:

$$\Delta = \frac{\partial c}{\partial S}$$

# Derivation of Black-Scholes-Merton equation (cont.)

- Inserting Δ one derives the following formula for dV:

$$dV = -\frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 dt - \frac{\partial c}{\partial t}dt$$

- Since the portfolio is risk free, it should earn the risk free rate over dt. Hence:

$$dV = V\,r\,dt$$

- Inserting this into the equation above, and rearranging the terms, one derives the Black-Scholes-Merton partial differential equation.

$$r\,S\frac{\partial c}{\partial S} - r\,c + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + \frac{\partial c}{\partial t} = 0$$

# Remarks

- The specifics of the derivative did not enter the derivation. Hence, the Black-Scholes-Merton equation holds for arbitrary derivatives.
- The above equation is a linear parabolic partial differential equation.
- Different derivatives imply different boundary conditions.
- The drift term $\mu$ does not show up in the Black-Scholes-Merton equation. Hence, for an investor, who is able to construct the risk free portfolio, the drift is unimportant, because it can be hedged away.

  In order to hedge the risk of the underlying security over the entire life time of the derivative, the investor has to adjust his holding of the stock after each time step. This is possible, because the strategy outlined above is self-financing. Because of this rebalancing the strategy is also called a dynamic hedge (which replicates the option).

# Remarks (cont.)

- If an investor, who is able to implement the dynamic hedge, exists, he will determine the value of the derivative, because whenever the value of the derivative deviates from the value implied by the Black-Scholes-Merton equation, arbitrage opportunities for the investor arise, and he is able to earn a risk free profit above the risk free rate. This action will automatically push the traded price of the derivative to its Black-Scholes value.

# Some simple examples

- It is easy to prove that the pricing formulas for the following two financial assets are consistent with the BSM-equation:

- the underlying $c = S$

- a bank account $c = e^{rt}$

- a forward $c = \left(Se^{r(T-t)} - K\right)e^{-r(T-t)} = \left(S - Ke^{-r(T-t)}\right)$

  proof

$$\frac{\partial c}{\partial S} = 1 \quad \frac{\partial^2 c}{\partial S^2} = 0 \quad \frac{\partial c}{\partial t} = -r\,K\,e^{-r(T-t)}$$

$$\Rightarrow r\,S - r\left(S - K\,e^{-r(T-t)}\right) - r\,K\,e^{-r(T-t)} = 0$$

# General formula for European style assets

- One can show that if the boundary condition of a financial asset at t'=T is given by f($S_T$,T) then the value at t'=t can be calculated by solving the integral:

$$f(S,t)=\frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}\sigma}\int_{-\infty}^{+\infty} d(\ln S')f(S',T)e^{-\frac{\left(\ln S+\left(r-\frac{\sigma^2}{2}\right)(T-t)-\ln S'\right)^2}{2\sigma^2(T-T)}}$$

- Note: American or Bermudan style options cannot be evaluated with this formula. Here specifying a boundary condition for t'=T is not sufficient.

# Pricing formulas for European calls and puts

- Boundary conditions for European call and put:

- call: $c(T) = \max(S(T) - K, 0)$
- put: $p(T) = \max(K - S(T), 0)$

$$\tilde{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy\, e^{-\frac{1}{2}y^2}$$ cumulative normal distribution

- Pricing formula: $c(t) = S(t)\tilde{N}(d_1) - K\, e^{-r(T-t)}\tilde{N}(d_2)$

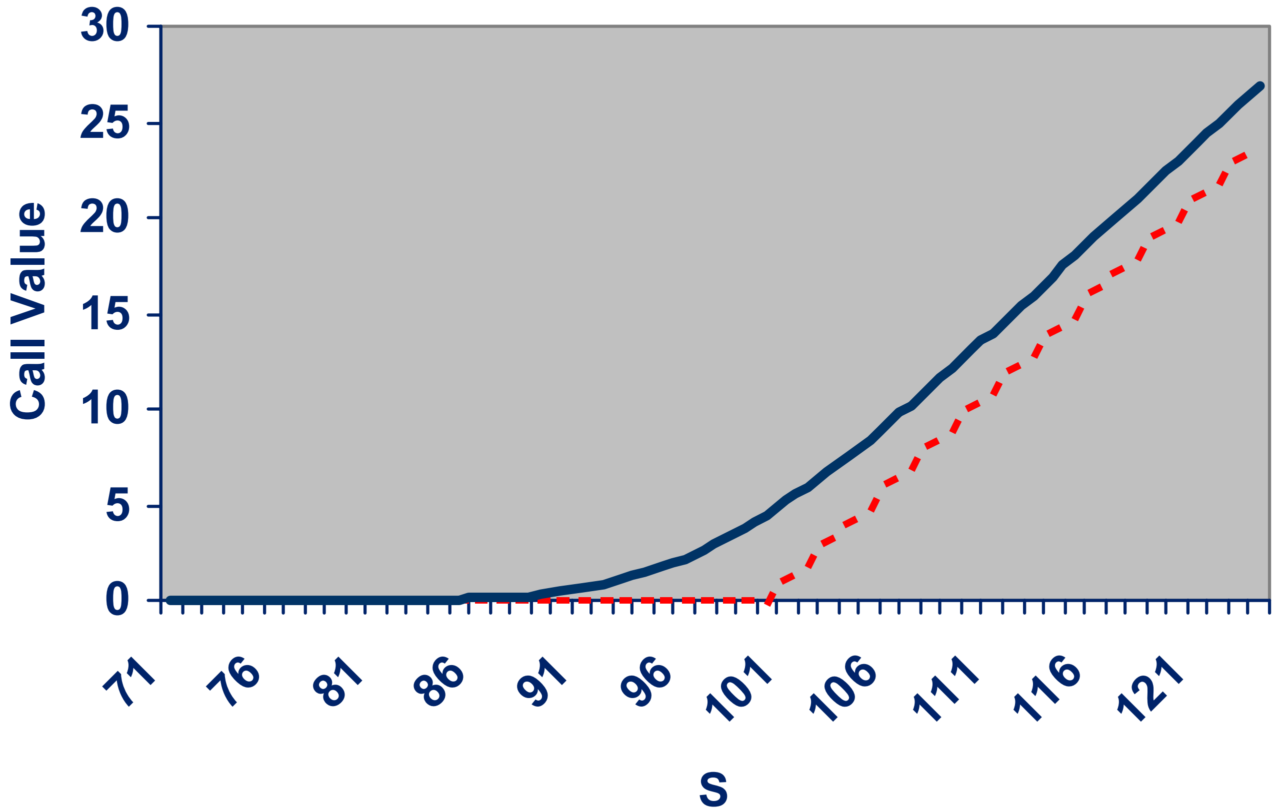
$$p(t) = K\, e^{-r(T-t)}\tilde{N}(-d_2) - S(t)\tilde{N}(-d_1)$$

- where:

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

# Example: European Call




**Call**

Strike = 100
Maturity = 0.5 years
Volatility = 10 %
r = 6 %

# Martingale measures and the pricing of options

- A martingale measure is a probability measure satisfying:

$$Z(t) = E(Z(T))$$

- It is possible to show that in a complete (hedges in the underlying instrument are possible) and arbitrage free market any financial contract normalized by a second financial contract follows a process with the martingale property. Hence:

$$Z = \frac{c}{V} \Rightarrow Z(t) = \tilde{E}\left(\frac{c(T)}{V(T)}\right)$$

risk neutral probability measure (not the real world probability measure)

numeraire good (normalizes the price)

# An example

- In our simplified model the bank account is the numeraire. Since it follows a deterministic process, we can move it before the expectation operator.

$$V(t)=1$$

$$\widetilde{E}(V(T))=V(T)=e^{r(T-t)}$$

$$\Rightarrow Z(t)=\frac{c(t)}{V(t)}=c(t)=\widetilde{E}\left(\frac{c(T)}{V(T)}\right)=e^{-r(T-t)}\widetilde{E}(c(T))$$

- Notice, however, in order to utilize the above equation we must apply the correct measure when calculating the expectation value. In our case this reduces to determining the drift of Ito process.
If you do this, you find that the drift must equal the short rate.

# Valuing European style options with Monte Carlo

- Since European style options depend only on the value of the underlying at maturity, it is not necessary to simulate the full path of the underlying during the lifetime of the option. A simulation of the underlying value at maturity is sufficient. Hence, the following process must be simulated:

$$S(T) = S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\int_t^T dz_t\right) = S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma Z\right)$$

drift has been set to short rate, because we have to use the risk neutral probability measure

- Where Z is a Gaussian variable:

$$Z \sim N\left(0, \sqrt{T-t}\right)$$

# Outline of Monte Carlo procedure

1. Draw a Gaussian distributed random number and calculate the value of the underlying at maturity.
2. Calculate the value of the financial contract from the underlying value.
3. Repeat steps 1 and 2 n times, where n should be sufficient large.
4. Calculate the average value of the financial contract at maturity.
5. Discount back the average value at maturity to obtain the present value of the contract.

# Hands On: European calls and puts

- Use the procedure outlined above to calculate the value of an European style call and an European style put.

  Compare the calculated value to the respective analytical value, and examine the effect of increasing the number of MC steps (calculate the quadratic derivation from the analytical value).
- Use the following parameters

$$t = 0$$
$$T = 0.5y, 1.0y$$
$$S(0) = 100$$
$$K = 80, 100, 120$$
$$r = 0.03$$
$$\sigma = 0.25$$

# Hands On: The Asymmetric Power Call

- The Asymmetric Power Call has the following payoff profile:

$$c(T) = \max\left(0, S(T)^n - K^n\right)$$

- Use MC to calculate the value for n=2 and 5.

  Again examine the convergence behavior. For this, calculate the “exact” value of the option by valuing it with a sufficient number of MC-steps (>50000).

- Note: Use the parameters from the first hands on.

# Hands On: The Straddle

- A straddle is an option strategy which has the following payoff profile:

$$c(T) = |S(T) - K|$$

- Once more examine the convergence behavior (note: since the straddle is a combination of a call and a put with identical strike and maturity, the analytical value can be calculated).

# Reducing the error in the estimate

- The speed of convergence of MC-methods with respect to N will always be of the order $O(N^{0.5})$.
- Hence, in order to increase the accuracy by one order of magnitude the number of samples has to increase by two orders of magnitude.
- However, a number of variance reduction techniques exist, which improve the accuracy of the result without requiring an increase in the number of random numbers used for the estimate.

# Antithetic Variables

- In the Antithetic Variable technique in each MC step two estimates of the option value are calculated.
- For the second estimate we use the same random number, however with a reversed sign.

$$\left.\begin{aligned} c_1 &= e^{-rT} f\left(S(Z,T),T\right) \\ c_2 &= e^{-rT} f\left(S(-Z,T),T\right) \end{aligned}\right\} \text{ use one drawing to calculate two estimates}$$

- Although the two values will be correlated the expected quadratic derivation of the estimated option value will be lower than if only one value is used.

# Antithetic Variables (cont.)

- The expected quadratic deviation of the option value calculated from the two estimates is given by:

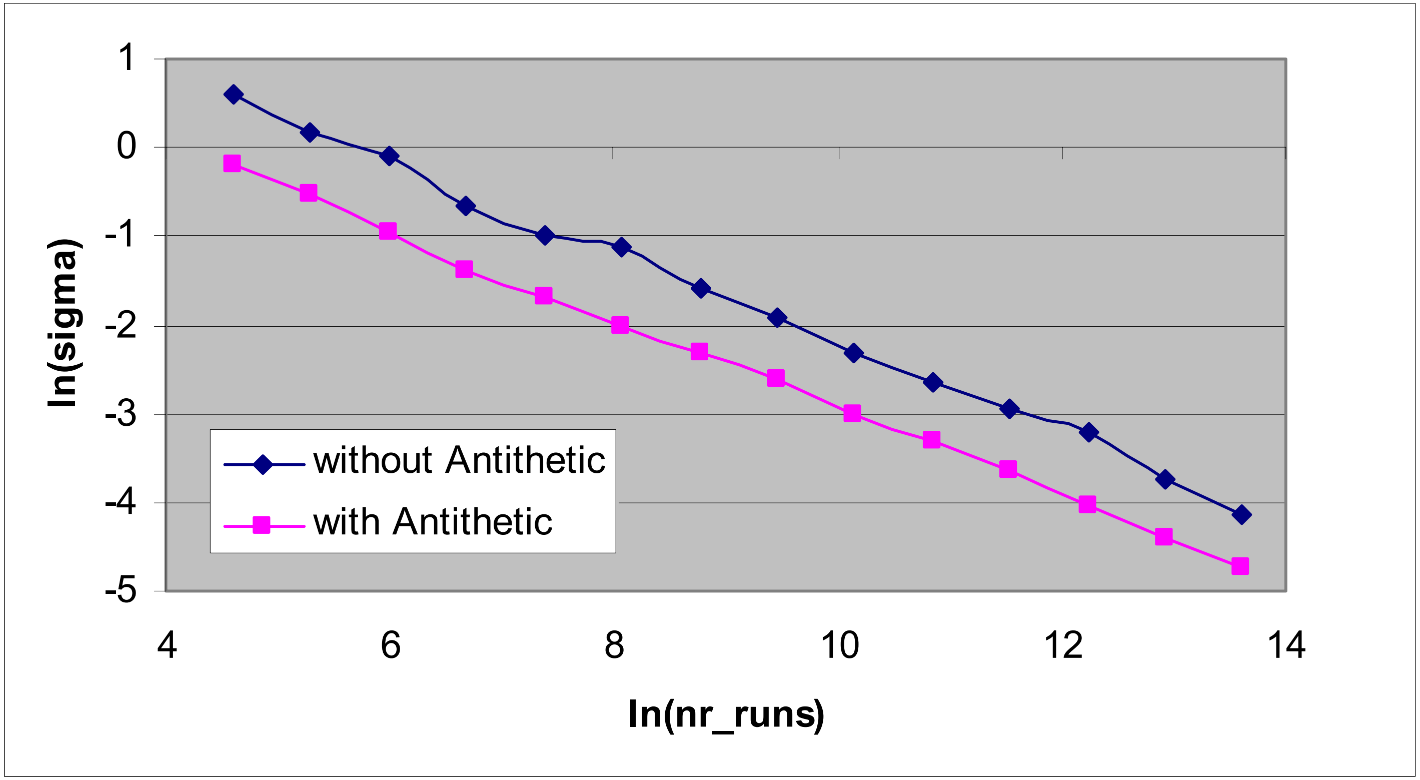
$$E\left(\left(\frac{1}{2}(c_1 + c_2) - c\right)^2\right) = E\left(\frac{1}{2}\left((c_1 - c)^2 + (c_2 - c)^2\right) - \frac{1}{4}(c_1 - c_2)^2\right)$$

$$= E\left((c_1 - c)^2\right) - \frac{1}{4}E\left((c_1 - c_2)^2\right) \leq E\left((c_1 - c)^2\right)$$

- Where c denotes the analytical value of the option.
- Whenever the payoff-profile of an option is asymmetric, the expected quadratic difference between the two estimates will be large.

# Example

At the money European Call valued using MC.




sigma = square-root of average deviation from BS-value (averaged over 100 runs)

# Hands-On

- Add the antithetic variable technique to your programs and redo the valuation of the options.
- Check that it really improves the convergence speed.
- For which option do you expect only mediocre results?

# Control Variate

- The Control Variate technique assumes that you have a second derivative with a similar payoff profile but for which an analytical formula is available.
- In the MC simulation one then not only calculates an estimate for the first but also for the second option. And then adds the difference between the estimated and analytical value of the second option to the estimated value of the first option.

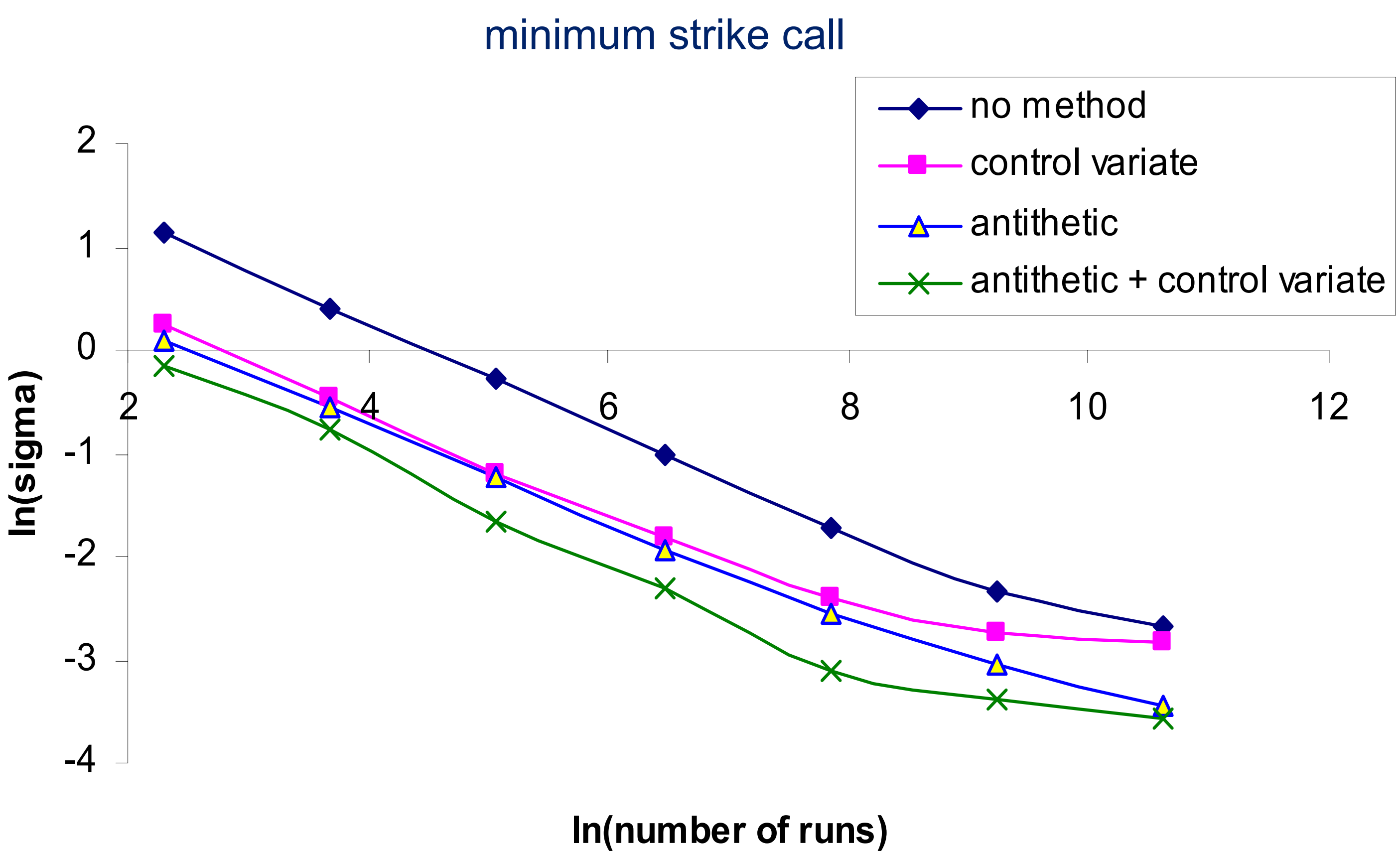
# Control Variate (cont.)

- Let
  - $c_1$= MC-estimate of the first option, for which no analytical formula is available
  - $c_2$= MC-estimate of the second option, for which an analytical formula exists
  - $C_2$= analytical value of the second option
- new estimate: $\tilde{c}_1 = c_1 - (c_2 - C_2)$

- Rationale: If the options are very similar the error in the estimate should be very similar as well.

However, options must be very similar so that the method leads to an improvement.

# example for application of Control Variate technique



# Hands-On

- Incorporate the control variate technique into your programs and redo the valuation.
  - ideally it should be possible to switch on and off the different speed up techniques from the command line
- Measure the improvement in convergence speed.

# Advanced Options

- Asian style and Look-Back options are path dependent, and therefore require the simulation of the entire path of the underlying during the option's lifetime. The density of the sampled path will depend on the specifics of the option contract.
  - examples:
    - Asian Style = options on the average
    - look back = options on the minimum or maximum value
- While for the valuation of European style options MC is often an "overkill", because the dimension of the problem is lower than four, the technique is ideally suited for this kind of options, because with every point in the sample path the dimensionality of the problem increases by one.

# Hands-On

Use MC to value the following options:

Asian Style:

- Average Price Call: $c(T) = \max(0, average(S(t')) - K)$
- Average Strike Call: $c(T) = \max(0, S(T) - average(S(t')))$

Look Back:

- Minimum Strike Call: $c(T) = S(T) - \min(S(t'))$
- Look Back Straddle: $c(T) = \max(S(t')) - \min(S(t'))$

- Note: Use the parameters from the first exercise. Also assume that the year consists of 250 trading days, hence:

$$\Delta t = \frac{1}{250}$$

# Risk

Remember that:

- The value of a portfolio depends on its stochastic underlying and is thus itself a stochastic variable.
- The expected future value of the portfolio can be calculated from process of the underlying variables.

What is risk from the investors point of view?

- Any **negative** deviation from the expected future or current (depends on definition) value of the portfolio.

# Value-at-Risk

- In the financial markets the most common quantity to measure the risk of a portfolio is the so called Value-at-Risk (VaR).
- Definition:

$$V = Portfolio\ Value$$

$$c = confidence\ level\ (one\ sided)$$

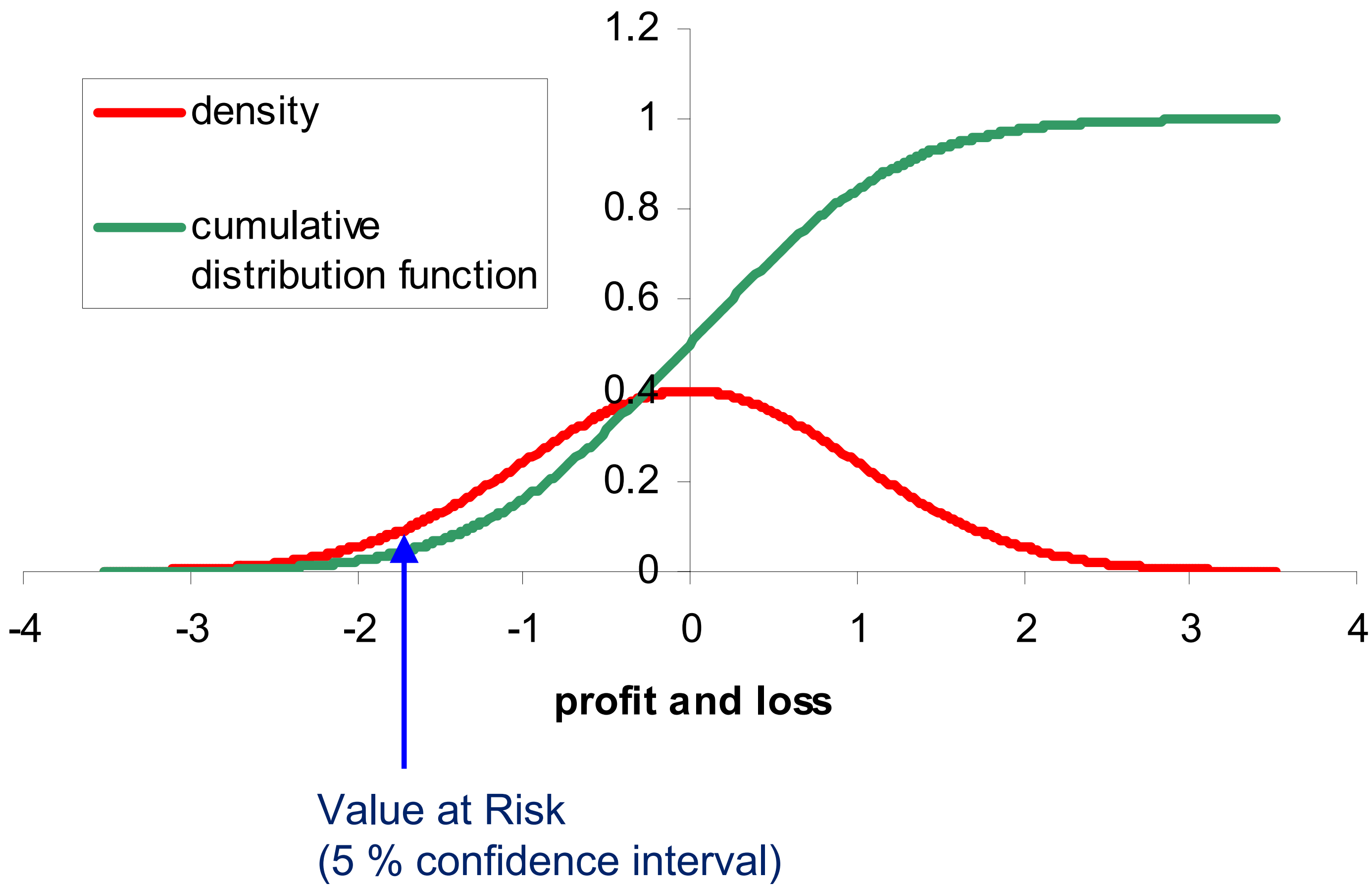
$$x = V(T) - V(t=0) = \Delta V = profit\ \&\ loss$$

$$p(x,T) = probability\ density\ of\ V$$

$$c = 1 - \int_{-VaR}^{+\infty} dx\ p(x,T) = \int_{-\infty}^{-VaR} dx\ p(x,T)$$

- Hence, to calculate the VaR you need a confidence level (usually one or five percent) and a time horizon (usually one or ten days).

# VaR


density
cumulative distribution function
1.2
1
0.8
0.6
0.4
0.2
0
-4
-3
-2
-1
0
1
2
3
4
profit and loss
Value at Risk
(5 % confidence interval)

# Example

- Suppose you own a portfolio of stocks and your VaR is 100 EUR.
  - one sided confidence interval of 5%
  - time horizon of one day
- Then on average at only one out of 20 trading days your daily loss will exceed 100 Euro.
  - Does not tell you, how much you will actually loose (expected loss above VaR-level is also called Excess-VaR).
  - Management often forgets that on average on one out of 20 trading days they will actually incur a loss exceeding their VaR (so if their current VaR-level scares them, they should close some of their positions).

# Remarks

- Analytical formula is only available for some special cases.
  - Approximations exist.
- VaR depends strongly on the stochastic process used for modeling the security prices.
  - Hence, any error in your model will lead to an error in your VaR-estimate.
  - Remember that no accurate and usable model exists.
- In the following we will use a one day time horizon:
  - drift becomes unimportant (remember, we cannot measure it anyway)
  - we can approximate the log-normal distribution by the normal distribution

# VaR for one linear security

- We first consider a portfolio which consists of only one stock. In this simple case we can calculate the VaR from the volatility of the stock (note: we use a 1 percent confidence interval).
- In this simple case we can use the following model:

$$\Delta z = \frac{\Delta S}{\sigma S} \sim N\left(0, \sqrt{\Delta T}\right)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dx \exp\left(-\frac{x^2}{2}\right)$$

$$\phi(-2.33) \approx 0.01$$

$$\Rightarrow VaR \approx 2.33 \sqrt{\Delta T} \sigma S$$

# VaR for a portfolio of two stocks

- We now consider a portfolio of two stocks. The value of the portfolio is given by:

$$V = n_1 S_1 + n_2 S_2$$

- While the change in value over a small time step is given by:

$$\Delta V = n_1 \Delta S_1 + n_2 \Delta S_2 = n_1 \sigma_1 S_1 \Delta z_1 + n_2 \sigma_2 S_2 \Delta z_2$$

- Since this is the sum of two Gaussian variables ΔV will also be a Gaussian variable, hence: $VaR = 2.33\sqrt{\Delta T}\sigma_V$
- Where:

$$\sigma_V = \sqrt{(n_1 \sigma_1 S_1)^2 + (n_2 \sigma_2 S_2)^2 + 2\rho_{12}(n_1 \sigma_1 S_1)(n_2 \sigma_2 S_2)}$$

# Arbitrary number of stocks

$$x_i = n_i\, S_i = Value\ of\ i-th\ position$$

$$\sigma_{ij} = \sigma_i\, \sigma_j\, \rho_{ij} = Covariance\ Matrix$$

$$z = \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} n_i\, S_i = Value\ of\ Portfolio$$

$$\Delta z = \sum_{i=1}^{N} \Delta x_i = \sum_{i=1}^{N} n_i\, \Delta S_i = Change\ in\ Value$$

Gaussian Variable, hence we can compute its VaR

$$\sigma_z^2 = \Delta T \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} = \Delta T \sum_{i=1}^{N} \sum_{j=1}^{N} n_i n_j S_i S_j \sigma_i \sigma_j \rho_{ij}$$

$$VaR = a\sigma_z$$

# Extension to non-linear instruments

$$x_i = n_i \, f_i\left(\{S_j\}, t\right) = Value\ of\ i-th\ position$$

$$\Delta x_i = n_i \sum_{j=1}^{N} \frac{\partial f_i}{\partial S_j} \Delta S_j$$

$$z = \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} n_i \, f_i = Value\ of\ Portfolio$$

$$\Delta z = \sum_{i=1}^{M} \Delta x_i \approx \sum_{i=1}^{M} n_i \sum_{j=1}^{N} \frac{\partial f_i}{\partial S_j} \Delta S_j = \sum_{j=1}^{N} \left( \sum_{i=1}^{M} n_i \frac{\partial f_i}{\partial S_j} \right) \Delta S_j$$

$$= \sum_{i=1}^{N} \frac{\partial z}{\partial S_i} \Delta S_i$$

$$\sigma_z^2 = \Delta T \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial z}{\partial S_i} \frac{\partial z}{\partial S_j} \sigma_i \sigma_j \rho_{ij}$$

$$VaR = a \sigma_z$$

# MC-VaR

1. Draw N samples from the multi-normal distribution of your n underlying variables.
   - Evaluate your portfolio and record the value.
2. Order the N simulated portfolio values from worst to best.
3. "Throw away" the N(1-confidence level) worst values.
4. The worst value now gives you your VaR.
   - Remember VaR=loss which is not exceeded (hence we do not need to do any interpolation)

however, if this matters, you should draw more samples

# Remarks

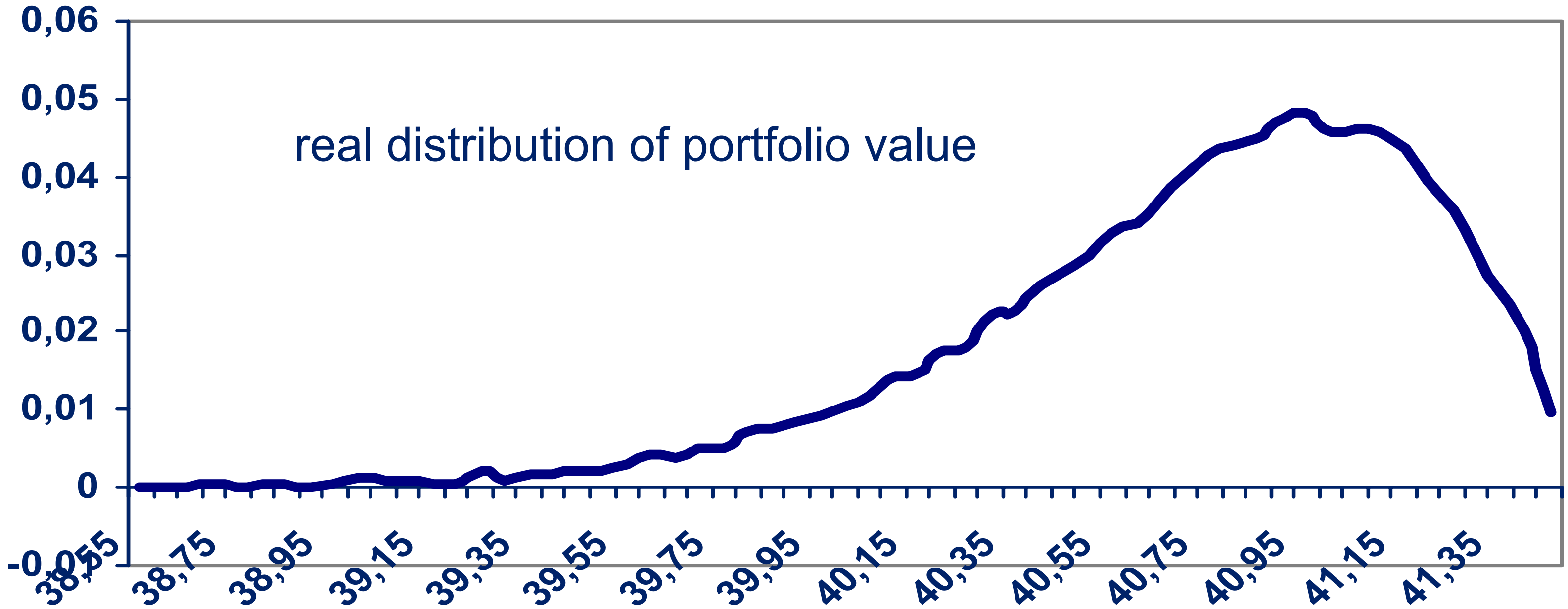
- In order to simulate n stochastic variables the root of the correlation matrix must be computed.
  - Cholesky-Decomposition
  - methods for dealing with degenerate matrices exist
- The problem with MC-VaR is that one is interested in the tail not the center of the distribution.
  - Importance sampling difficult to achieve, since for non-linear portfolios the VaR limit can be anywhere in the phase space.
  - Most commercial software packages use brute force methods.
- MC- and linear VaR can give very different results.
  - linear VaR is not the method of choice for highly non-linear portfolios

# Example: A hedged portfolio

$$V(S,t) = -f(S,t) + \Delta S \quad \Rightarrow \quad \Delta_V = \partial_S V = -\partial_S f + \Delta = 0$$

$$\Rightarrow \quad VaR_{lin} = 0$$

linear approximation predicts
no variance in portfolio value



# Literature

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  - excellent introduction to interest rate models
- RiskMetrics, “Technical Documents”
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  - download from website www.riskmetrics.com
- P. Jäckel, "Monte Carlo methods in Finance"
  - short but great