Way above the efficient frontier

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Risk and Return

- *Portfolio management* is about gaining as much return as possible by keeping the risk under control.
  - *return* is the *expected* return of the portfolio over the *next* holding period.
  - *risk* is also an estimator covering the *next* holding period.

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1.1 Return

- Portfolio consisting of holdings $N_k, k = 1, \dots M$ in M financial instruments (*assets*) with asset values V_k dependent on n market parameters (*risk factors*) $S_i, i = 1, \dots, n$

$$V(t) = \sum_{k=1}^{M} N_k V_k(\mathbf{S}(t))$$

- Portfolio return over a time period δt

$$r_V(\mathbf{S}(t))\delta t = \sum_{k=1}^{M} w_k(\mathbf{S}(t)) r_k(\mathbf{S}(t))\delta t \tag{1.1}$$

  – with position weights w_k and instrument and portfolio returns r_V and r_k.

$$r_V \delta t := \frac{\delta V}{V}\ , \quad r_k \delta t := \frac{\delta V_k}{V_k}\ , \quad w_k := \frac{N_k V_k}{V} \quad \text{for } k = 1, \dots M \tag{1.2}$$

  – in vector notation

$$r_V = \mathbf{w}^T \mathbf{r}$$

- The same holds for the *expected* portfolio return R_V

$$R_V = \mathrm{E}\left[r_V\right] = \mathbf{w}^T\mathbf{R} \tag{1.3}$$

  - $\mathbf{R}$ denotes the vector of *expected* asset returns R_k for $k = 1, \ldots, M$.

$$\mathbf{R} = \mathrm{E}\left[\mathbf{r}\right] \quad \Longleftrightarrow \quad R_k = \mathrm{E}\left[r_k\right] \quad \text{for} \quad k = 1, \ldots, M$$

1.2 Risk

- Portfolio value change induced by market parameter changes (delta normal approximation)

$$\delta V \approx \sum_{k=1}^{M} N_k \sum_{i}^{n} \frac{\partial V_k}{\partial S_i}\delta S_i = \sum_{k=1}^{M} N_k \sum_{i}^{n} \Delta_i^k \delta S_i$$

- Variance of the portfolio value change

$$
\begin{aligned}
\operatorname{var}[\delta V] &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{M} N_k \Delta_i^k S_i \delta\Sigma_{ij} N_l \Delta_j^l S_j \\
&= V^2 \sum_{k,l=1}^{M} \sum_{i,j=1}^{n} w_k \Omega_i^k \delta\Sigma_{ij} \Omega_j^l w_l \qquad (1.4) \\
&= V^2 \mathbf{w}^T \delta\mathbf{C}\mathbf{w}
\end{aligned}
$$

  – with the risk factor covariance matrix

$$\delta\Sigma_{ij} = \operatorname{cov}[\delta \ln S_i, \delta \ln S_j] = \sigma_i \rho_{ij} \sigma_j \delta t \quad , \quad i, j = 1, \ldots n$$

  – and the asset covariance matrix $\delta\mathbf{C}$ with matrix elements

$$\delta C_{kl} = \operatorname{cov}[\delta \ln V_k, \delta \ln V_l] \approx \sum_{i,j=1}^{n} \Omega_i^k \delta\Sigma_{ij} \Omega_j^l \quad , \quad k, l = 1, \ldots M \qquad (1.5)$$

  – usually considered: investments in the *risk factors* themselves (e.g., in stocks)

– here: investments in *instruments* (e.g., derivatives)

– The only difference (in the delta normal approximation): instead of $\delta\mathbf{\Sigma}$ use $\delta\mathbf{C}$ as the covariance matrix.

- Delta normal *Value at Risk*[1]:

$$\begin{aligned}\mathrm{VaR}_V(c) &\approx |Q_{1-c}|\sqrt{\mathrm{var}\,[\delta V]} \\ &= |Q_{1-c}|\,V\sqrt{\mathbf{w}^T\delta\mathbf{C}\mathbf{w}} \\ &= |Q_{1-c}|\,V\sqrt{\delta t}\sqrt{\sum_{k,l=1}^{M} w_k w_l \sum_{i,j=1}^{n} \Omega_i^k \sigma_i \sigma_j \rho_{ij} \Omega_j^l}\end{aligned} \tag{1.6}$$

– proportional to the portfolio value V

– much better suited for optimization: Value at Risk in percentage terms: $\mathrm{VaR}_V(c)/V$

[1] Deutsch, H.-P., *Derivatives and Internal Models*, 3rd Edition, Palgrave, London 2004, ISBN 1-4039-2150-4, for a review of the 2nd Edition in RISK Magazine see http://www.d-fine.de/pool/bibliothek/hpd_book_review.pdf

– Also: arbitrary but constant factors $|Q_{1-c}|$ and $\sqrt{\delta t}$ irrelevant for optimization.

$$\frac{\mathrm{VaR}_V(c)}{V\ \ |Q_{1-c}|\ \sqrt{\delta t}} = \frac{1}{\sqrt{\delta t}}\sqrt{\frac{1}{V^2}\mathrm{var}\left[\delta V\right]} = \frac{1}{\sqrt{\delta t}}\sqrt{\mathrm{var}\left[\frac{\delta V}{V}\right]} = \frac{1}{\sqrt{\delta t}}\sqrt{\sigma_V^2 \delta t} = \sigma_V$$

- Thus, our risk measure is simply the portfolio volatility

$$\sigma_V = \frac{1}{\sqrt{\delta t}}\sqrt{\mathbf{w}^T \delta \mathbf{C} \mathbf{w}} =: \sqrt{\mathbf{w}^T \mathbf{C} \mathbf{w}} \tag{1.7}$$

– with the new matrix $\mathbf{C}$ defined as

$$\mathbf{C} := \frac{1}{\delta t}\delta \mathbf{C} \tag{1.8}$$

$$C_{kl} = \sum_{i,j=1}^{n} \Omega_i^k \sigma_i \rho_{ij} \sigma_j \Omega_j^l \quad , \quad k,l = 1,\ldots M$$

- We can just as well use σ_V^2 as the risk measure for portfolio optimization purposes

$$\sigma_V^2 = \mathbf{w}^T \mathbf{C} \mathbf{w} \tag{1.9}$$

– This and Eq. 1.3 are the starting point for classical *mean variance optimization* and *Markovitz theory*.

2

Classical Portfolio Optimization

- Consider portfolios *fully invested* in risky assets

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- All of the capital is invested in the M financial instruments

$$\sum_{k=1}^{M} w_k = 1 \tag{2.1}$$

$$\mathbf{w}^T\mathbf{1} = \mathbf{1}^T\mathbf{w} = 1$$

2.1 The Minimum Risk Portfolio

- Minimize the portfolio risk by varying the position weights w_k
  - using the Lagrange Method
- The w_k can well be negative if short selling is allowed.
- The only constraint is to be fully invested, i.e., Eq. 2.1.
- The Lagrange function is

$$\mathcal{L} = \underbrace{\mathbf{w}^T\mathbf{C}\mathbf{w}}_{\text{to be minimized}} - \lambda\underbrace{\left[\mathbf{w}^T\mathbf{1} - 1\right]}_{\text{constraint}}$$

– extremal $\mathcal{L}$ for

$$\mathbf{0} \stackrel{!}{=} \frac{\partial \mathcal{L}}{\partial \mathbf{w}^T} = 2\mathbf{C}\mathbf{w} - \lambda \mathbf{1} \tag{2.2}$$

– minimal $\mathcal{L}$ for

$$\frac{\partial^2 \mathcal{L}}{\partial \mathbf{w} \partial \mathbf{w}^T} = 2\mathbf{C} > \mathbf{0}$$

∗ the assets' covariance matrix $\mathbf{C}$ must be positive definite (usually no problem), i.e., it's inverse $\mathbf{C}^{-1}$ exists

- Solution of the optimization

$$\mathbf{w} = \frac{\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{C}^{-1}\mathbf{1}} \tag{2.3}$$

– These weights depend only on the assets' covariances but *not* on their returns.

- Risk and return of this portfolio are

$$\sigma^2_{\min} = \mathbf{w}^T \mathbf{C} \mathbf{w} = \frac{1}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}} \tag{2.4}$$
$$R_{\min} = \mathbf{w}^T \mathbf{R} = \frac{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{R}}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}}$$

2.2 The Efficient Frontier

- Again find portfolio with minimal risk
  - but observe one *more* constraint: the portfolio must have a *given* fixed expected return R:

$$R_V \stackrel{!}{=} R \Longleftrightarrow \mathbf{w}^T \mathbf{R} \stackrel{!}{=} R \tag{2.5}$$

- The Lagrange Function to be minimzed is

$$\mathcal{L} = \underbrace{\mathbf{w}^T \mathbf{C} \mathbf{w}}_{\text{To be mimized}} - \lambda_1 \underbrace{\left[\mathbf{w}^T \mathbf{R} - R\right]}_{\text{Constraint}} - \lambda_2 \underbrace{\left[\mathbf{w}^T \mathbf{1} - 1\right]}_{\text{Constraint}}$$

- Solution of the optimization: weights of an efficient portfolio

$$\mathbf{w} = \sigma^2_{\min}\mathbf{C}^{-1}\mathbf{1} + \frac{\sigma^2_V - \sigma^2_{\min}}{R - R_{\min}}\mathbf{C}^{-1}\left(\mathbf{R} - \mathbf{1}R_{\min}\right) \tag{2.6}$$

  – for each given portfolio return R there is one *unique* portfolio which minimizes the risk

- Risk of an efficient portfolio

$$\sigma^2_V = \sigma^2_{\min} + \frac{\left(R - R_{\min}\right)^2}{\mathbf{R}^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{R} - R^2_{\min}/\sigma^2_{\min}} \tag{2.7}$$

  – This minimal portfolio variance for a given R as a function of R is a parabola.
  – The R as a function of the minimal portfolio volatility σ_V is called the *efficient frontier*.

2.3 The Market Portfolio and the Sharpe Ratio

- Goal of any investment
  - Maximize what you get for the risk you take
  - In other words: maximizes the *market price of risk*
  - In other words: maximizes the *Sharpe Ratio*

$$\gamma := \frac{R - r_f}{\sigma} \tag{2.8}$$

- The *optimal* or *market portfolio* is the fully invested portfolio with maximum Sharpe Ratio
  - it must lie on the efficient frontier (because fully invested)
  - solving Eq. 2.7 for R yields the Sharpe Ratio for any portfolio on the efficient frontier

$$\gamma = \frac{R_{\min} - r_f}{\sigma} \pm \sqrt{1 - \sigma_{\min}^2/\sigma^2}\sqrt{\mathbf{R}^T\mathbf{C}^{-1}\mathbf{R} - R_{\min}^2/\sigma_{\min}^2}$$

- To find the *maximal* Sharpe Ratio move along the efficient frontier by varying σ

$$0 = \frac{\partial \gamma}{\partial \sigma} = r_f - R_{\min} + \sigma_{\min}^2 \frac{\sqrt{\mathbf{R}^T \mathbf{C}^{-1} \mathbf{R} - R_{\min}^2 / \sigma_{\min}^2}}{\sqrt{\sigma^2 - \sigma_{\min}^2}} \tag{2.9}$$

  - Solving this for σ^2 yields the variance of the market portfolio

$$\sigma_m^2 = \sigma_{\min}^2 + \sigma_{\min}^4 \frac{\mathbf{R}^T \mathbf{C}^{-1} \mathbf{R} - R_{\min}^2 / \sigma_{\min}^2}{(R_{\min} - r_f)^2} \tag{2.10}$$

  - The return R_m of the market portfolio follows as

$$R_m = R_{\min} + \sigma_{\min}^2 \frac{\mathbf{R}^T \mathbf{C}^{-1} \mathbf{R} - R_{\min}^2 / \sigma_{\min}^2}{R_{\min} - r_f} \tag{2.11}$$

- The weights of the optimal portfolio are:

$$\begin{aligned} \mathbf{w}_m &= \mathbf{w}_{\min} + \sigma_{\min}^2 \mathbf{C}^{-1} \frac{\mathbf{R} - \mathbf{1} R_{\min}}{R_{\min} - r_f} \qquad (2.12) \\ &= \frac{\mathbf{C}^{-1} \widehat{\mathbf{R}}}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \end{aligned}$$

– where a "hat" denotes *excess* returns, i.e.

$$\widehat{\mathbf{R}} = \mathbf{R} - \mathbf{1}r_f \quad , \quad \widehat{R}_V = R_V - r_f \text{ , etc.}$$

2.4 The Capital Market Line

- Why is the market portfolio so important?
- The market portfolio has unique risk σ_m and unique return R_m.
- In particular, the risk of the market portfolio cannot be chosen by the investor.
- Always invest in the market portfolio, even if σ_m doesn't coincide with your risk preference σ_{required}.
  - If $\sigma_{\text{required}} < \sigma_m$: invest a percentage w of the total capital in the market portfolio and the rest risk free.
  - If $\sigma_{\text{required}} > \sigma_m$: borrow money and invest the total sum in the market portfolio.

- This strategy has the same (i.e., maximal) Sharpe Ratio as the market portfolio.
  - Investing or borrowing in the money market doesn't produce neither risk nor excess return.

- The return and risk of this strategy are

$$\begin{aligned} R_V &= wR_m + (1-w)\, r_f \\ \sigma_V &= w\sigma_m \end{aligned} \tag{2.13}$$

- Write the return as

$$\begin{aligned} R_V &= r_f + (R_m - r_f)\frac{\sigma_V}{\sigma_m} \\ &= r_f + \gamma_m \sigma_V \quad \text{with} \quad \gamma_m = \frac{R_m - r_f}{\sigma_m} \end{aligned} \tag{2.14}$$

Conclusion 1 *The best possible return as a function of the risk taken is a straight line with slope given by the Sharpe Ratio of the market portfolio.*

- This straight line is called the *capital market line*.
- Eq. 2.14 can be read in the following way:
  - For each unit of additional risk σ_V an investor is willing to take, the expected return increases by γ_m.
  - The Sharpe Ratio is therefore the *market price of risk* of the investment universe consisting of the M risky assets (and of course the money market account).
    * In complete agreement with the market price of risk encountered in option pricing
    * i.e. with the drift appearing in the Girsanov Theorem for the transition from the real world measure to the martingale measure used for pricing

Summary 2 ***The Sharpe Ratio*** γ_m ***of the market portfolio for an investment universe consisting of*** M ***risky assets is the slope of the capital market line describing the expected return of the optimal investment strategy as a function of investment risk. The Sharpe Ratio is therefore the market price of risk for this investment universe.***

- Therefore, everybody should always invest only in the market portfolio and the money market.
  - What, if all of the above breaks down ....?

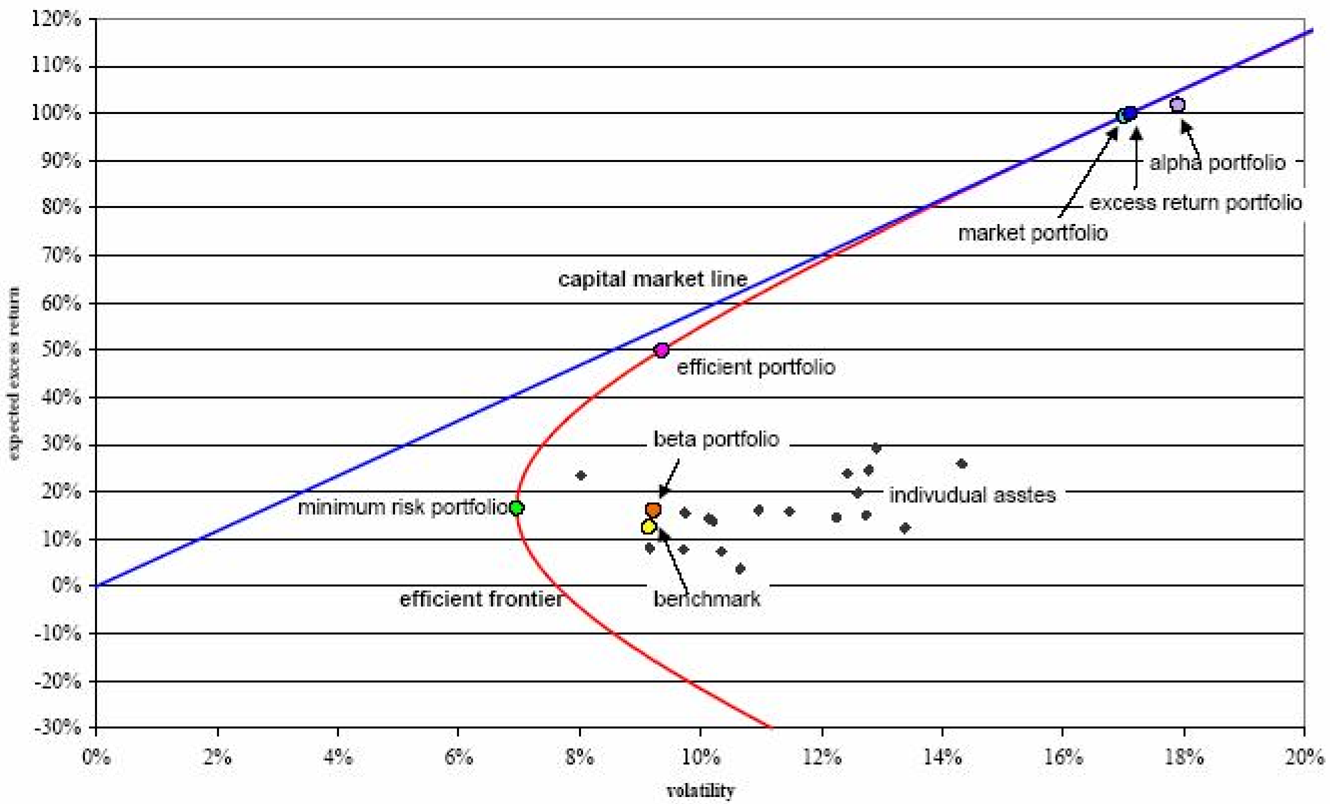

Figure 2.1: A risk return diagram. The "assets" are the DJ STOXX 600 sector indices in 1996.

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The Condition for a Valid Market Portfolio

- Since $\sigma_m^2 \geq \sigma_{\min}^2$ we should also have $R_m \geq R_{\min}$ in Eq. 2.11, since otherwise:
  - Sharpe Ratio of minimal risk portfolio > Sharpe Ratio of the market portfolio,

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– **contradiction** to the market portfolio having *maximal* Sharpe Ratio.

- Thus, we must have

$$\sigma^2_{\min} \frac{\mathbf{R}^T \mathbf{C}^{-1} \mathbf{R} - R^2_{\min}/\sigma^2_{\min}}{R_{\min} - r_f} \overset{!}{\geq} 0 \tag{3.1}$$

- Observe that[1]

$$\mathbf{R}^T\mathbf{C}^{-1}\mathbf{R} \geq \frac{\left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{R}\right)^2}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} = R_{\min}^2/\sigma_{\min}^2$$

- Thus, $\mathbf{R}^T\mathbf{C}^{-1}\mathbf{R} - R_{\min}^2/\sigma_{\min}^2 \geq 0$.

[1] To show this, we need the *Cauchy-Schwarz inequality* for a scalar product of two arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\left(\mathbf{x}^T\mathbf{y}\right)^2 \leq \left(\mathbf{x}^T\mathbf{x}\right)\left(\mathbf{y}^T\mathbf{y}\right)$$

Since $\mathbf{C}^{-1}$ is positive definite, the inequality is preserved if "multiplied" by $\mathbf{C}^{-1}$ twice:

$$\left(\mathbf{x}^T\mathbf{C}^{-1}\mathbf{y}\right)^2 \leq \left(\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right)\left(\mathbf{y}^T\mathbf{C}^{-1}\mathbf{y}\right)$$

Now choose $\mathbf{x} = \mathbf{1}$ and $\mathbf{y} = \mathbf{R}$:

$$\left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{R}\right)^2 \leq \left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}\right)\left(\mathbf{R}^T\mathbf{C}^{-1}\mathbf{R}\right)$$

$$\frac{\left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{R}\right)^2}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} \leq \mathbf{R}^T\mathbf{C}^{-1}\mathbf{R}$$

- Since $\sigma^2_{\min} \geq 0$ (as for any squared number), Condition 3.1 is only fulfilled for

$$R_{\min} \geq r_f \tag{3.2}$$

- Usually, $R_{\min} > r_f$ holds since:
  - The minimum risk portfolio is a portfolio of *risky* assets.
  - Therefore, the market should require an expected return *above* the risk free rate.

- However, Eq. 3.2 "sometimes" (e.g., in the years 2000, 2001 and 2002) does not hold. Then:
  - The expected return of the market portfolio, Eq. 2.11, is lower than $R_{\min}$.
  - Thus, the market portfolio cannot have maximum Sharpe Ratio.
  - But the market portfolio still has extreme Sharpe Ratio (Eq. 2.9 still holds.)
  - The extremum is now the *minimum* and the market portfolio is the *worst* investment possible.

Conclusion 3 *If the return of the fully invested minimum risk portfolio is less than the risk free rate, then there is no* fully invested *portfolio with maximum Sharpe Ratio.*

- Intuitively: If markets are turning down then
  - Any portfolio which is net long cannot be the best choice.
  - By definition, a fully invested portfolio is 100% net long.
  - All portfolios on the efficient frontier are fully invested.
  - Therefore, the whole efficient frontier does *not* contain any good portfolio!
  - There are many portfolios *above* the efficient frontier.

4

Attributes and Characteristic Portfolios

- The assets in a portfolio have many *characteristics* or *attributes*
  - expected return, market capitalization, beta with respect to an index, membership in a certain economic sector, etc.

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- The *exposure* of a portfolio V to a certain attribute $\mathbf{a}$ of the assets is defined as

$$a_V \equiv \sum_{k=1}^{M} w_k a_k = \mathbf{w}^T \mathbf{a} \tag{4.1}$$

  - Examples:
    * If the attributes a_i for $i = 1, \ldots M$ measure how strongly the assets belong to the automotive industry then a_V is the exposure of the portfolio to the automotive industry.
    * If the characteristics a_i are the asset returns R_i then the exposure a_V is simply the portfolio return R_V.

- The *characteristic portfolio* for an attribute $\mathbf{a}$ is defined as the minimum risk portfolio with unit exposure to that attribute, i.e., with

$$\mathbf{w}^T \mathbf{a} = 1 \tag{4.2}$$

- No other constraints. In particular: *no* fully invested constraint.

- The characteristic portfolio minimizes its risk while observing Constraint 4.2. The Lagrange function is

$$\mathcal{L} = \underbrace{\mathbf{w}_a^T \mathbf{C} \mathbf{w}_a}_{\text{To be Minimized}} - \lambda \underbrace{\left[\mathbf{w}_a^T \mathbf{a} - 1\right]}_{\text{Constraint}}$$

- Solving this optimization problem yields the weights, variance and excess return of the characteristic portfolio

$$\mathbf{w}_a = \frac{\mathbf{C}^{-1}\mathbf{a}}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}} \tag{4.3}$$

$$\sigma_a^2 = \frac{1}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}}$$

$$\widehat{R}_a = \frac{\mathbf{a}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}}$$

- The Sharpe Ratio of the characteristic portfolio is

$$\gamma_a \equiv \frac{\widehat{R}_a}{\sigma_a} = \frac{\mathbf{a}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\sqrt{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}}} \tag{4.4}$$

- A useful relation between two characteristic portfolios

  - Let b_a denote the exposure of a characteristic portfolio V_a of an attribute $\mathbf{a}$ to *another* attribute $\mathbf{b}$
  - Likewise, a_b denotes the exposure of characteristic portfolio V_b to the attribute $\mathbf{a}$
  - The relation between those two exposures is the same as between the variances of the two characteristic portfolios:

$$\frac{b_a}{a_b} = \frac{\sigma_a^2}{\sigma_b^2} \tag{4.5}$$

4.1 The Leverage

- Let's choose the attribute vector $\mathbf{L}$ to be

$$\mathbf{L} = \mathbf{1} \Longleftrightarrow L_k = 1 \quad , \quad k = 1, \ldots M \tag{4.6}$$

- The exposure of *any* portfolio V to this attribute is the *extent of investment* in risky assets, also called *leverage*

$$L_V = \mathbf{w}^T\mathbf{L} = \mathbf{w}^T\mathbf{1} = \sum_{k=1}^{M} w_k \tag{4.7}$$

- A fully invested portfolio has $L_V = 1$.

- The characteristic portfolio V_L for this attribute is the fully invested portfolio with minimum risk.

  – From Eq. 4.3, this portfolio has the following weights, variance and excess return

$$\mathbf{w}_L = \frac{\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}\ , \quad \sigma_L^2 = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} \tag{4.8}$$

$$\widehat{R}_L = \frac{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}$$

  ∗ In full accordance with Eqs 2.3 and 2.4.

4.2 The Excess Return

- Let's now choose the attribute vector to be the assets' *expected excess returns*

$$\mathbf{A} = \widehat{\mathbf{R}} = \mathbf{R} - r_f \mathbf{1} \tag{4.9}$$

- The exposure of *any* portfolio V to this attribute is the portfolio's expected excess return:

$$A_V = \mathbf{w}^T \widehat{\mathbf{R}} = \sum_{k=1}^{M} w_k \left(R_k - r_f \right) \tag{4.10}$$

- The *characteristic* portfolio V_A for this attribute is the *minimum risk* portfolio with unit exposure, i.e., with excess return equal to 1.

$$\widehat{R}_A \equiv R_A - r_f \overset{!}{=} 1 \tag{4.11}$$

  - According to Eq. 4.3, this the characteristic portfolio has weights and variance:

$$\mathbf{w}_A = \frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{R}^{\mathbf{T}}\mathbf{C}^{-1}\widehat{\mathbf{R}}} \quad , \quad \sigma_A^2 = \frac{1}{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \tag{4.12}$$

- This characteristic portfolio has minimal risk for a *given*, i.e., *constant* excess return
  - Therefore, it maximizes the ratio $1/\sigma_A = \widehat{R}_A/\sigma_A$.
  - But this is the portfolio's *Sharpe Ratio*.

Conclusion 4 ***The characteristic portfolio for attribute $\mathbf{A} = \widehat{\mathbf{R}}$ has maximal Sharpe Ratio. This holds for any situation, even if Inequality 3.2 does not hold.***

$$\gamma_{\max} = \gamma_A = \frac{1}{\sigma_A} = \sqrt{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \tag{4.13}$$

- For any positive number λ, an investment with weights $\lambda \cdot w_k$ for $k = 1, \ldots M$, will have the *same* Sharpe Ratio.
  - All portfolios with maximal Sharpe Ratio (but different leverage via different λ) lie on a straight line with slope $\gamma_{\max}$.
  - We call this line the *Capital Market Line*.

4.2.1 Comparing your Portfolio with the Best

- To relate the excess return of *your* portfolio V (having weights $\mathbf{w}$) with this characteristic portfolio solve Eq. 4.12 for $\widehat{\mathbf{R}}$

$$\mathbf{w}^T \widehat{\mathbf{R}} = \frac{\mathbf{w}^T \mathbf{C} \mathbf{w}_A}{\sigma_A^2}$$

- The *Sharpe Ratio* of your portfolio is:

$$\gamma_V = \frac{\mathbf{w}^T \widehat{\mathbf{R}}}{\sigma_V} = \frac{\mathbf{w}^T \mathbf{C} \mathbf{w}_A}{\sigma_V \sigma_A} \gamma_A$$

- The factor in front of γ_A is the correlation between the two portfolios, because:

$$\begin{aligned} \mathbf{w}^T \mathbf{C} \mathbf{w}_A &= \sum_{k,i=1}^{M} w_k w_{A,i} \delta t \operatorname{cov}(r_k, r_i) \\ &= \delta t \operatorname{cov}(r_V, r_A) \\ &= \operatorname{corr}(r_V, r_A) \sigma_V \sigma_A \end{aligned}$$

Conclusion 5 *The Sharpe Ratio of your portfolio is the maximal Sharpe Ratio times the correlation between your portfolio and the characteristic portfolio V_A.*

$$\gamma_V = \rho_{V,A}\ \gamma_A \quad \text{for all portfolios } V \tag{4.14}$$

Conclusion 6 *Only portfolios fully correlated with the characteristic portfolio V_A have maximal Sharpe Ratio.*

4.3 The Market Portfolio Revisited

- The characteristic portfolio for $A = \widehat{\mathbf{R}}$ has an (enforced) excess return of 1, i.e., of 100%.
  - Therefore, it usually contains significant leverage.
- This leverage is simply the exposure of portfolio V_A to the attribute $\mathbf{L}$

$$L_A = \mathbf{w}_A^T \mathbf{1} = \mathbf{w}_A^T \mathbf{L}$$

- Eq. 4.5 applied to the attributes $\mathbf{L}$ and A directly yields the leverage

$$L_A = \frac{\sigma_A^2}{\sigma_L^2} A_L = \frac{\sigma_A^2}{\sigma_L^2} \underbrace{\mathbf{w}_L^T \widehat{\mathbf{R}}}_{\widehat{R}_L} = \frac{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \tag{4.15}$$

- Construct a portfolio V_m with leverage one (i.e., a *fully invested* portfolio), by simply dividing all the weights $\mathbf{w}_A$ by L_A:

$$\mathbf{w}_m = \frac{1}{L_A} \mathbf{w}_A = \frac{\sigma_L^2}{\sigma_A^2 \widehat{R}_L} \mathbf{w}_A \tag{4.16}$$

- With our results for σ_A^2, σ_L^2, $\widehat{R}_L$ and $\mathbf{w}_A$ we get

$$\mathbf{w}_m = \frac{\mathbf{C}^{-1} \widehat{\mathbf{R}}}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \tag{4.17}$$

  – Full accordance with Eq. 2.12!

Conclusion 7 *We have recovered the market portfolio using a cunning combination of two minimum risk portfolios: the minimum risk portfolio with leverage one and the minimum risk portfolio with excess return one!*

- It can be shown, that the market portfolio is itself a *characteristic* portfolio for the following attribute vector:

$$\mathbf{m} = \frac{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \widehat{\mathbf{R}}$$

- With the above weights, the market portfolio has the following excess return and variance

$$\widehat{R}_m = \widehat{\mathbf{R}}^T \mathbf{w}_m = \frac{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \tag{4.18}$$

$$\sigma_m^2 = \mathbf{w}_m^T \mathbf{C} \mathbf{w}_m = \frac{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\left(\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}\right)^2}$$

- From this, the Sharpe Ratio of the market portfolio can be written explicitly as

$$\begin{aligned}\gamma_m &\equiv \frac{\widehat{R}_m}{\sigma_m} \\ &= \frac{\left|\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}\right|}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \sqrt{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \\ &= \frac{\left|\widehat{R}_L\right|}{\widehat{R}_L} \gamma_A\end{aligned}$$

  - We explicitly see here, that this is the Sharpe Ratio of the characteristic portfolio V_A **or its negative**, depending on the expected return of the minimum risk portfolio being above the risk free rate or not!

Conclusion 8 *To be on the safe side in all situations, one should always use the characteristic portfolio for the attribute* $\widehat{\mathbf{R}}$ *to construct the Capital Market Line.*

4.4 The Efficient Frontier Revisited

- It can be shown that all portfolios on the efficient frontier are linear combinations of the two fully invested *characteristic* portfolios

$$\mathbf{w}_e = \frac{\widehat{R} - \widehat{R}_L}{\widehat{R}_m - \widehat{R}_L}\mathbf{w}_m + \frac{\widehat{R}_m - \widehat{R}}{\widehat{R}_m - \widehat{R}_L}\mathbf{w}_L \tag{4.19}$$

Conclusion 9 ***An efficient frontier portfolio with required excess return $\widehat{R}$ is simply the linear interpolation between the fully invested minimum risk portfolio and the market portfolio!***

- It can also be shown that each efficient frontier portfolio is itself a *characteristic* portfolio for the following attribute

$$\mathbf{e} = \frac{\sigma_L^2}{\sigma_e^2}\left(\frac{\widehat{R}_m - \widehat{R}}{\widehat{R}_m - \widehat{R}_L}\mathbf{1} + \frac{\widehat{R} - \widehat{R}_L}{\widehat{R}_m - \widehat{R}_L}\,\frac{\widehat{\mathbf{R}}}{\widehat{R}_L}\right) \tag{4.20}$$

- It can also be shown that asymptotically, for $\sigma_V \longrightarrow \infty$, the efficient frontier approaches two straight lines with slopes $\pm\sqrt{\gamma_m^2 - \gamma_L^2}$.

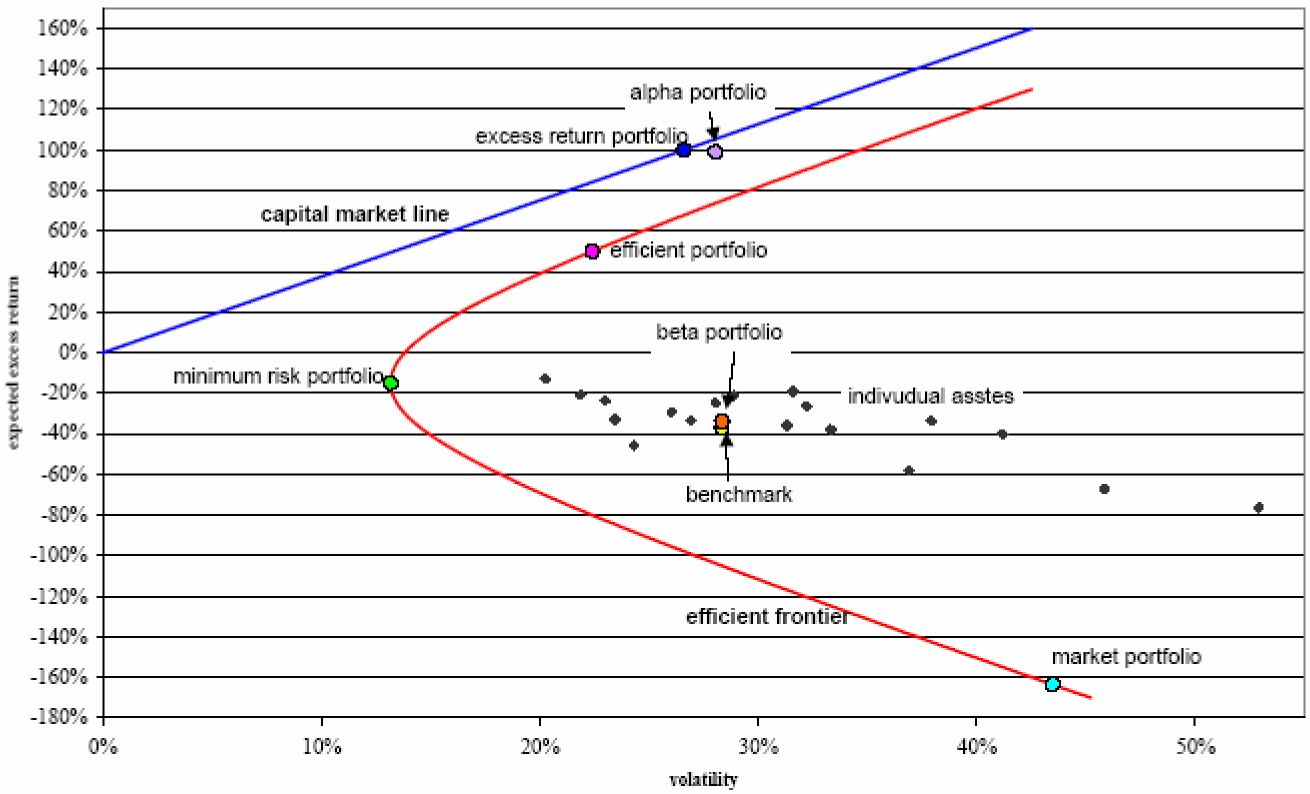

Figure 4.1: A risk return diagram. The "assets" are the DJ STOXX 600 sector indices in 2002.

5

Summary

- – You can be way above the efficient frontier!
  – We can do even better, far beyond Markovitz.
  – Read my book![1]
  – Observe a live portfolio at **www.triple-alpha.de**

[1] H.-P. Deutsch, *Quantitative Portfoliosteuerung*, Schäffer-Poeschel Verlag, Stuttgart 2005

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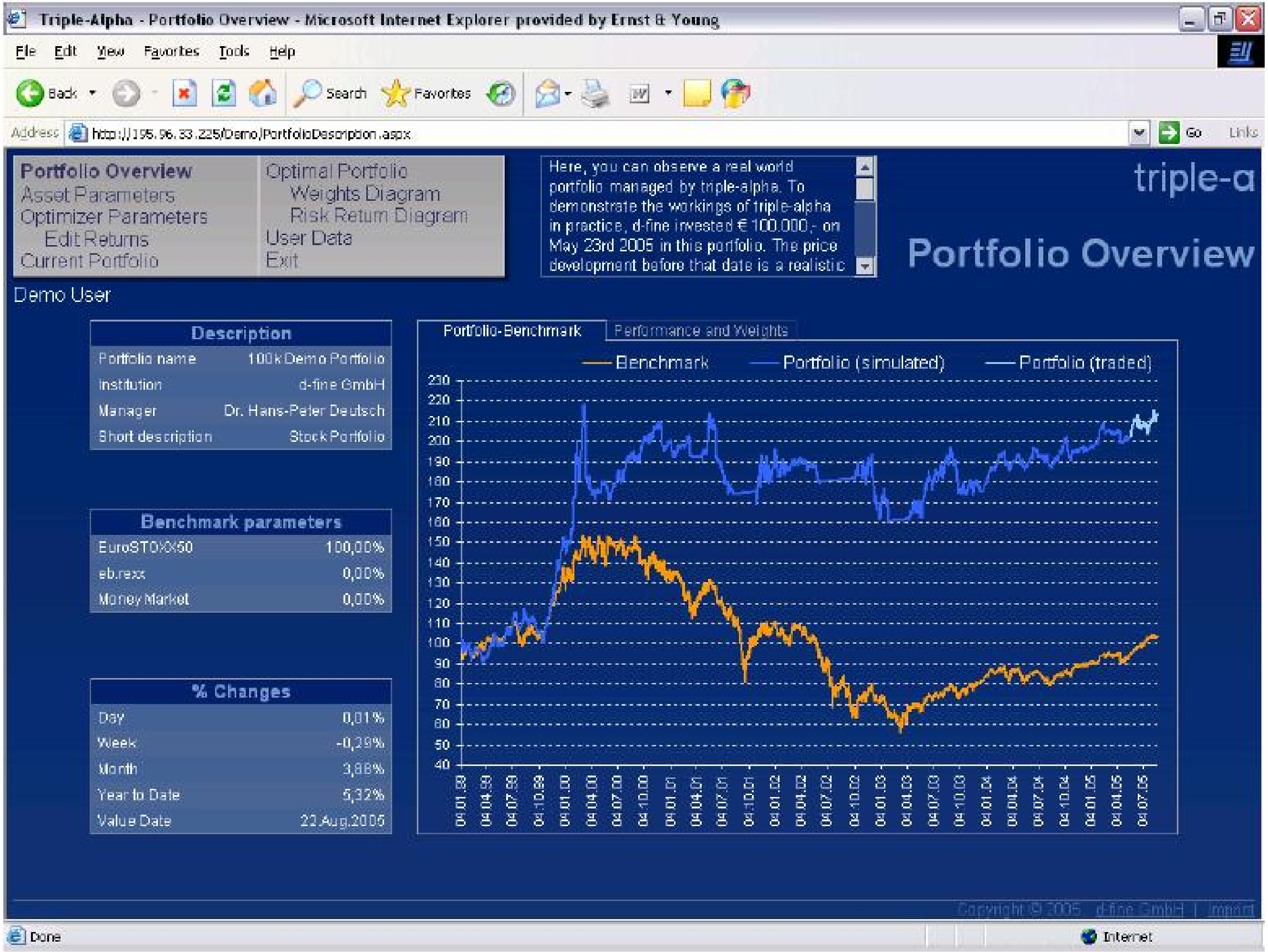

Triple-Alpha - Portfolio Overview - Microsoft Internet Explorer provided by Ernst & Young
File Edit View Favorites Tools Help
Back Search Favorites
Address http://195.96.33.225/Demo/PortfolioDescription.aspx
Go Links
Portfolio Overview
Asset Parameters
Optimizer Parameters
Edit Returns
Current Portfolio
Optimal Portfolio
Weights Diagram
Risk Return Diagram
User Data
Exit
Here, you can observe a real world portfolio managed by triple-alpha. To demonstrate the workings of triple-alpha in practice, d-fine invested € 100.000,- on May 23rd 2005 in this portfolio. The price development before that date is a realistic
triple-a
Portfolio Overview
Demo User
Description
Portfolio name 100k Demo Portfolio
Institution d-fine GmbH
Manager Dr. Hans-Peter Deutsch
Short description Stock Portfolio
Benchmark parameters
EuroSTOXX50 100,00%
eb.rexx 0,00%
Money Market 0,00%
% Changes
Day 0,01%
Week -0,29%
Month 3,88%
Year to Date 5,32%
Value Date 22.Aug.2005
Portfolio-Benchmark
Performance and Weights
Benchmark
Portfolio (simulated)
Portfolio (traded)
230
220
210
200
190
180
170
160
150
140
130
120
110
100
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- triple-α allows you to actively manage your portfolio in a risk-adjusted way based on objective mathematical-statistical methods.
- triple-α portfolios generally lie well above (in the worst case on) the famous “efficient frontier”.
- State of the art parameter-free risk measures are used with *no* model assumptions whatsoever.
- Advanced, adaptive Stopp-Loss strategies.
- All transaction costs are taken into account in a consistent risk-adjusted way and are factored into the investment decision in a mathematically correct manner.
- The optimisation is therefore realistic and not just an academic exercise in an idealised world.

For details, go online on www.triple-alpha.de

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