Valuation of multi-callable defaultable bonds

Marcus Evans conferences – Hybrid Products

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London, 7 Feb 2006

Introduction

Introduction

- Callable defaultable coupon bonds bear interest rate risk as well as credit default risk
  - E.g. corporate bonds
- Standard approach
  - Non-callable: Discounted cash flows with risky curve
  - Single call right: Black’76 (with adjustments for credit default risk)
  - Multiple call rights: some more advanced interest rate model, credit default risk is often ignored
- Problems:
  - Credit default risk is not taken fully into account
  - Inconsistencies with credit and interest rate derivates

How to solve the problems?

- Do I need a more advanced model? What features should the more advanced model have?
- What is the impact of spread volatility and correlation?
- What is the link between bond spreads and expected recovery rates?
- How do I get my bond spread risk in line with my CDS spread risk?
- What market data do I use?
- How can I extend the new approach to more complex, but defaultable, fixed income structures?

PV of a default-free coupon bond

$$CB(t) = X \sum_{t_k > t}^{N} \tau_n B(t, t_n) + B(t, t_N)$$

$$= XA(t, t_k, t_N) + B(t, t_N) \qquad t_{k-1} \le t < t_k$$

$CB(t)$	Price of a default-free coupon bond as of t
τ_n	Year fraction for nth payment
X	Fixed coupon rate
$A(t,t_k,t_N)$	Annuity including coupons from t_k to t_N as of t
$B(t,t_n)$	Price of a default-free discount bond as of t maturing at t_n

Defaultable bonds

Default-free versus defaultable zero bonds

Default-free zero bond

$$B(t,T) = e^{-\int_t^T f(t,s)ds}$$

Defaultable zero bond

$$B^d(t,T) = e^{-\int_t^T \left(f(t,s)+\lambda(t,s)\right)ds}$$

$f(t,T)$ Instantaneous forward rate at time T as of t

$\lambda(t,T)$ Instantaneous credit spread at time T (hazard rate)

Instantaneous credit spreads versus default probabilities

- Survival and default probability are functions of hazard rate $\lambda(t,T)$

$$1 - P^d(t,T) = P(t,T) = e^{-\int_t^T \lambda(t,s)ds}$$

$P^d(t,T)$ Probability as of t that obligor will default at some $\tau < T$

- The default probability density function (at time t) is

$$\lambda(t,T)\, e^{-\int_t^T \lambda(t,s)ds}$$

The simple spread model for a defaultable coupon bond

$$CB^d(t) = A^d(t, t_k, t_N)X + B^d(t, t_N)$$

$CB^d(t)$	Price of a defaultable coupon bond as of t
$A^d(t,t_k,t_N)$	Defaultable annuity including coupons from t_k to t_N as of t

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Assumptions in the basic spread model

- The risk free interest rate and the time of default are statistically independent
- There is no recovery (total loss given default LGD=100%)
- The term structure of defaultable zero bonds at time t is known

➔ Common approach to include spread risk in bond pricing

Introducing recovery

PV of a defaultable coupon bond with recovery

$$CB^d(t) = X A^d(t, t_k, t_N) + B^d(t, t_N) + \int_t^{t_N} R(s) B(t,s) \underbrace{\lambda(t,s) e^{-\int_t^s \lambda(t,r)dr}}_{\text{default probability density}} ds$$

$R(s)$ Recovery given default at future time $s>t$

The recovery term is the expected PV of the recovery value.

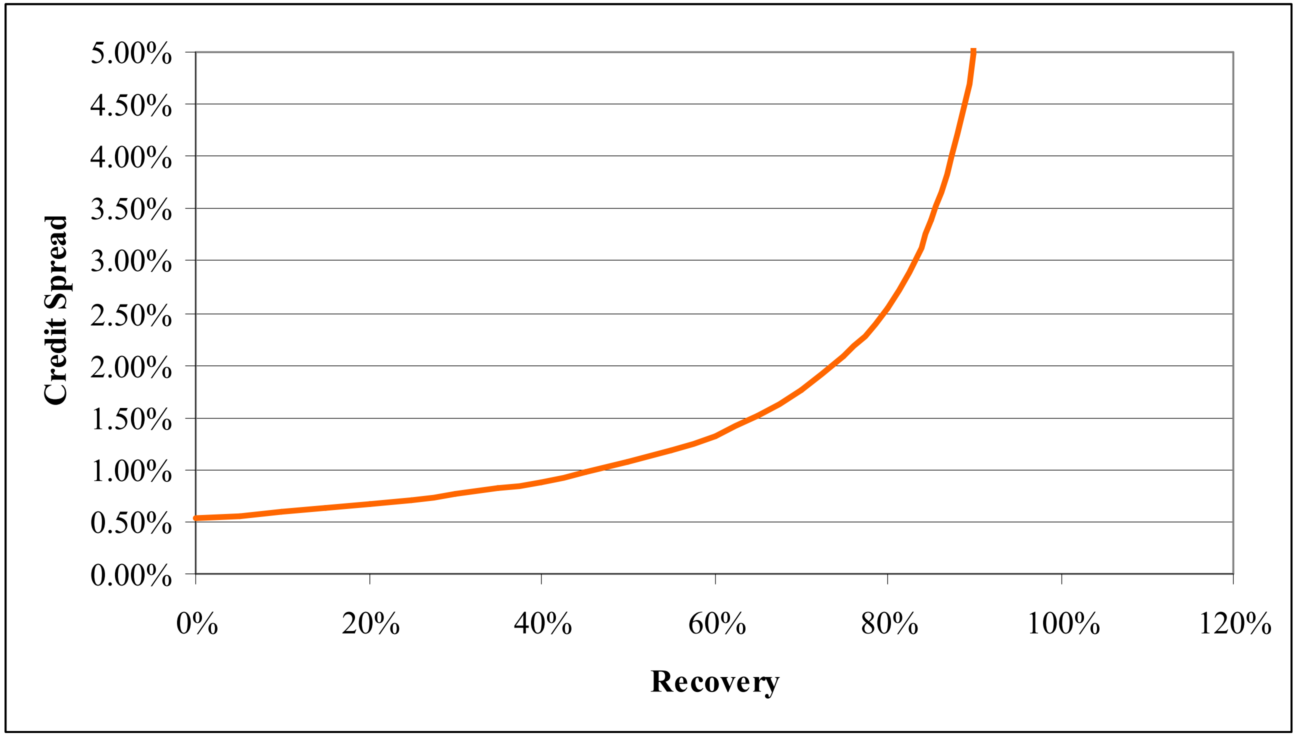
Basic assumption: deterministic recovery rate

Recovery impacts credit spreads

- More precisely, implied credit spreads are strictly increasing with the recovery rate

Bond maturity 10y
Coupon 4.5%
Flat interest rate 4%

Bond price fixed at $1



Defaultable bond options

PV of a risky bond call option using Black'76

- Standard approach: Black'76 on bond forward price

$$CBCO^{d}(t,T) = P(t,T)\,\mathrm{Black}^{\mathrm{C}}(F_P^d, K, r, \sigma_P, t, T)$$

$$F_P^d = X A^d(T, t_e, t_N) + B^d(T, t_N) \qquad t_{e-1} \le T < t_e$$

$CBCO^d$	PV of call option on coupon bond
T	Option expiry
K	Bond strike price
F_P^d	Forward cash bond price
r	Risk-free interest rate
σ_P	Price volatility of the coupon bond

PV of a risky bond put option using Black'76

- Counterparty is identical to issuer of coupon bond

$$CBPO^d(t,T) = P(t,T)\,\mathrm{Black}^{\mathrm{P}}(F_P^d, K, r, \sigma_P, t, T)$$

- Third party option writer

$$CBPO^d(t,T) = P^{3\mathrm{rd}}(t,T)\,\mathrm{Black}^{\mathrm{P}}(\widetilde{F}_P^d, K, r, \sigma_P, t, T)$$

$P^{3\mathrm{rd}}(t,T)$ Survival probability of the third party option writer

$\widetilde{F}_P^d$ Forward price of defaultable bond taking into account that bond issuer might have been defaulted before T

Valuation of options on risky bonds

- Assume deterministic hazard rate
- Replace risk free interest rate curve by risky interest rate curve
- Proceed with option valuation as for default free bonds

$$CBO^{d}(t,T) = \mathrm{Black}(F_P^{d}, K, r^{d}, \sigma_P, t, T)$$

- Problems with this approach:
  - Default intensities are not deterministic
  - Recovery effects are neglected
  - Asymmetry with respect to long/short, call/put, 3rd party issuers

Risky bond call option modelled as European swaption

- Use equivalence of swaptions and bond options for pricing

$$CBSRO^d(t,T) = A^d(t,t_e,t_N)\left(X\Phi\left(-d+\sigma\sqrt{T-t}\right)-F^d\Phi(-d)\right)$$

$$d = \frac{\ln(F^d/X)-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}$$

$$F^d = \frac{1-B^d(T,t_N)}{A^d(T,t_e,t_N)}$$

F^d	Defaultable forward par bond yield
σ	Swaption volatility
$\Phi(x)$	Cumulative normal distribution

Defaultable forward par bond yield with recovery

- Add correction term to defaultable forward par bond yield

$$\widetilde{F}^d = F^d - \frac{1}{A^d(T, t_e, t_N)} \int_T^{t_N} R(s) B^d(T, s) \lambda(s) ds$$

$\widetilde{F}^d$ Defaultable forward par bond yield including recovery

- In case of constant hazard rate and flat interest rate curve:

$$\widetilde{F}^d = F^d \left(1 - R \frac{\lambda}{r + \lambda} \right)$$

Advantages of the European swaption approach

- Interest rate risk factors are consistent for all interest rate derivatives
  - No interest rate volatility risk factors for every single issuer/bond
- The hazard rate and credit spread framework is consistent with usual credit derivative market conventions
  - Specific credit spread add-ons for bonds and bond derivatives will still be necessary due to liquidity differences and other effects

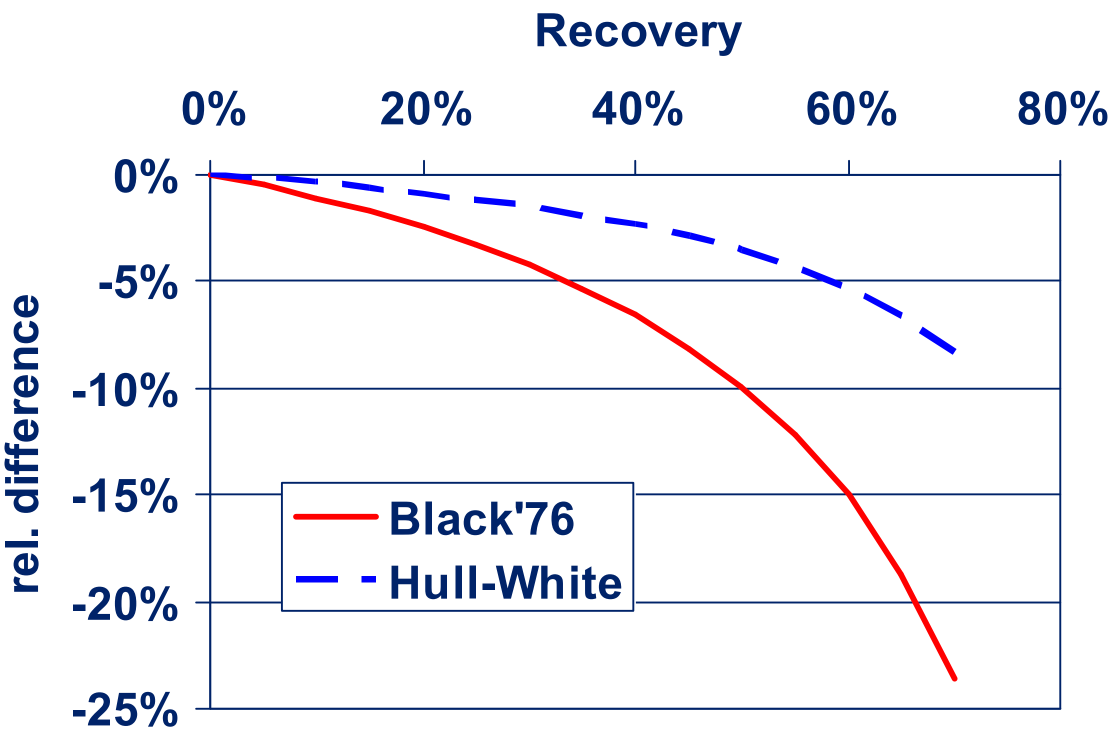
➔ Under this approach it is possible to work with overall consistent assumptions and models

Impact of recovery

- Impact of different recovery rates on bond option price if the bond price is kept constant

Bond Maturity 10y
Coupon 5.3%

Bond price fix at 1



Joint modelling of interest and hazard rate

Stochastic hazard rate models

- Assume hazard rate $\lambda(t)$ to be stochastic
- The local conditional hazard rate $\lambda(t)$ (or default intensity) describes the dynamics of the instantaneous credit spread

Example of a Hull-White-type stochastic hazard rate model:

$$d\lambda(t) = \mu_\lambda dt + \sigma_\lambda(t)\, dW_\lambda(t)$$

$$\mu_\lambda = \theta_\lambda(t) - a_\lambda \lambda(t)$$

a_λ Mean reversion of the hazard rate process
θ_λ Deterministic function of spread curve and volatility

Model with two Hull-White processes (I)

- Interest rate (short rate) process

$$dr(t) = \mu_r dt + \sigma_r(t)\, dW_r(t)$$

$$\mu_r = \theta_r(t) - a_r r(t)$$

a_r Mean reversion of short rate process

θ_r Deterministic function of interest rate curve and volatility

- Hazard rate process

$$d\lambda(t) = \left(\theta_\lambda(t) - a_\lambda \lambda(t)\right)dt$$
$$+\sigma_\lambda(t)\left(\rho^2\, dW_r(t) + \left(1-\rho^2\right)dW_\lambda(t)\right)$$

Model with two Hull-White processes (II)

Partial differential equation

$$\frac{\partial}{\partial t}V(t,r,\lambda) = \mu_r \frac{\partial}{\partial r}V(t,r,\lambda) + \frac{1}{2}\sigma_r^2 \frac{\partial^2}{\partial r^2}V(t,r,\lambda)$$
$$+ \mu_\lambda \frac{\partial}{\partial \lambda}V(t,r,\lambda) + \frac{1}{2}\sigma_\lambda^2 \frac{\partial^2}{\partial \lambda^2}V(t,r,\lambda)$$
$$+ \rho \frac{\partial}{\partial \lambda}\frac{\partial}{\partial r}V(t,r,\lambda) + (r+\lambda)V(t,r,\lambda) + R\lambda + f$$

V	Fair value of contigency claim
R	Expected (constant) recovery amount of V
f	Cash flow stream

Some remarks

- Model includes correlation of hazard and interest rate
- Extension to dynamic recovery rate fairly simple
- Estimated recovery rate depends on type of product
  - Options will have a significantly different recovery rate than bonds

Calibration of model parameters

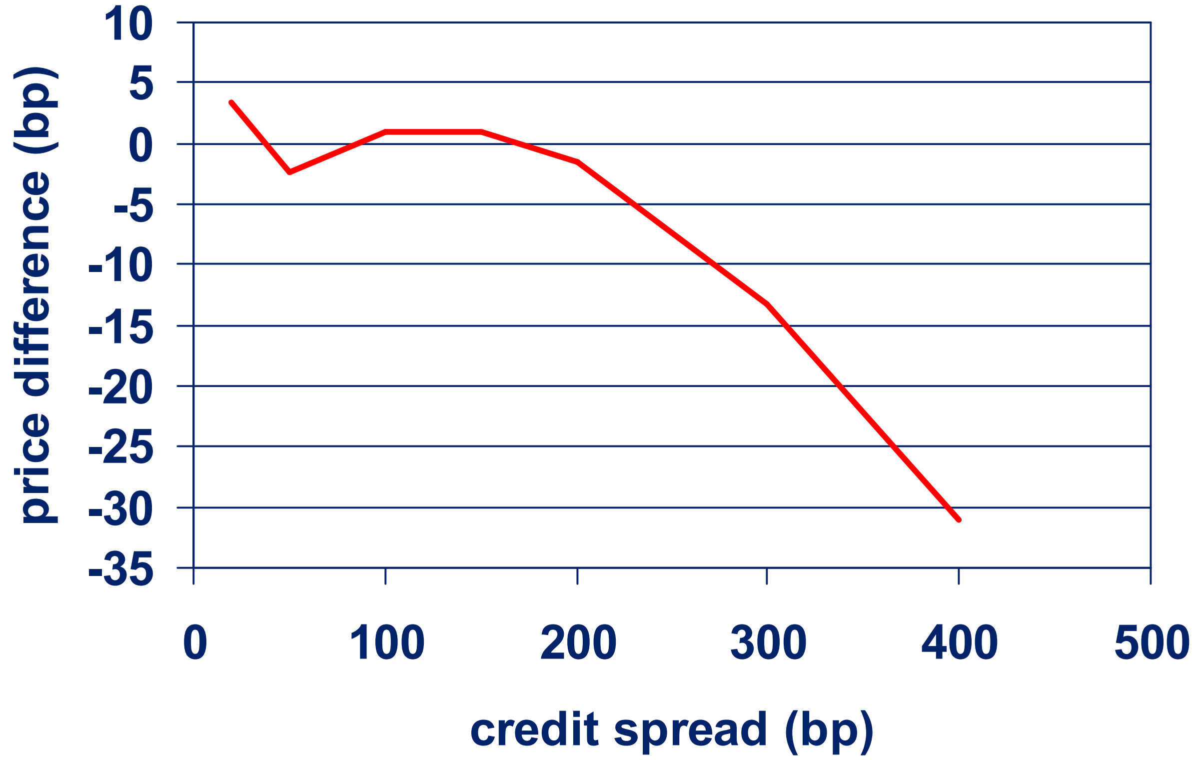
- Calibrate a_r and σ_r to swaption volatilities/prices
- Calculate θ_λ and θ_r based on credit spread curve and hazard rate volatility term structure (once it is given) resp. interest rate curve and short rate volatility term structure
- Calibration of a_λ and σ_λ is difficult
  - If available, calibrate to credit spread options (CSO)
  - Historical estimates
  - Mapping of available data for similar credit names
- Calibration of ρ and R is even worse
  - Historical data for specific credit name is in general not available
  - Implied calculations are instable

Difference between 2-Factor Model and Black'76

- Price difference in bp as a function of the spread

Bond Maturity 10y
Recovery 40%
Expiry 1y

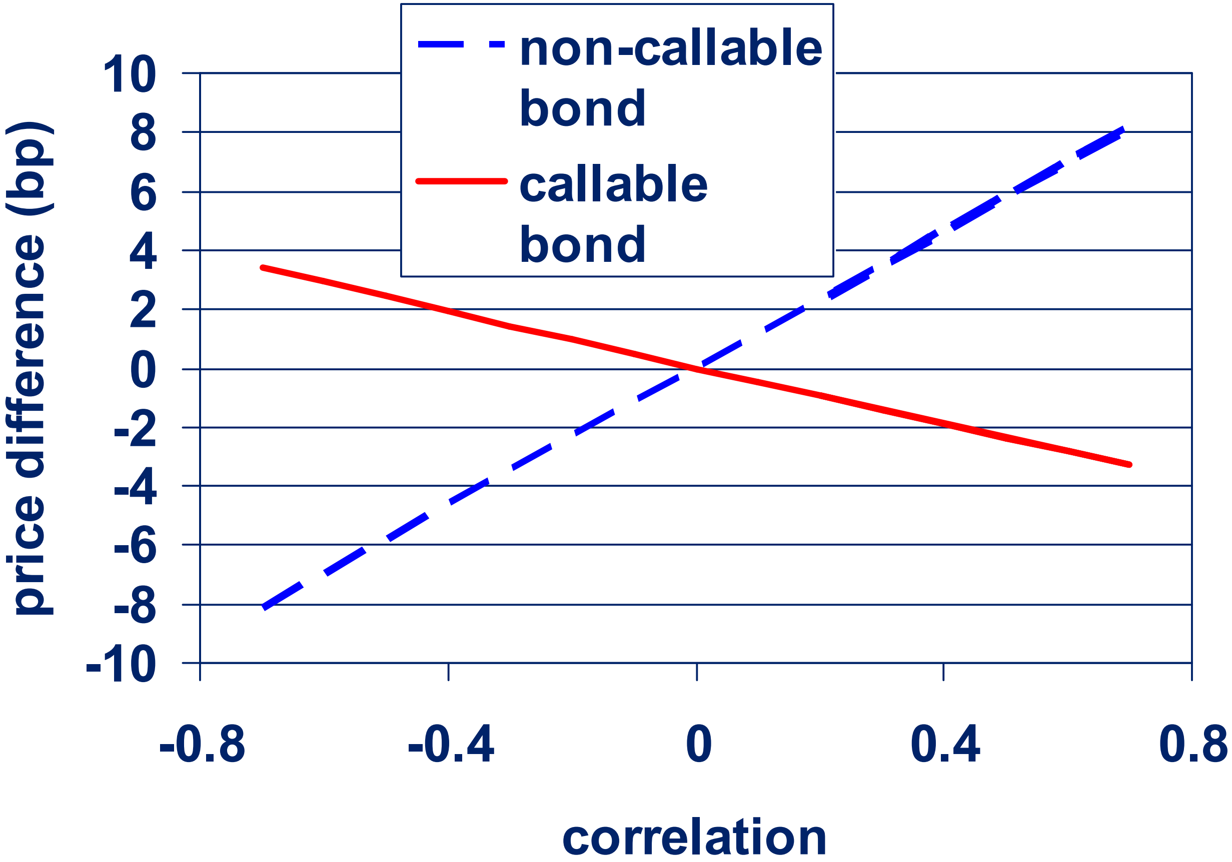
Spread volatility:
50% of spread



Impact of Cross Term (Correlation)

- Dependence of bond prices on correlation

Bond Maturity 10y
Coupon 5.37%
Spread volatility 0.2%
Spread 40 bp
Recovery 40%
Expiry 1y

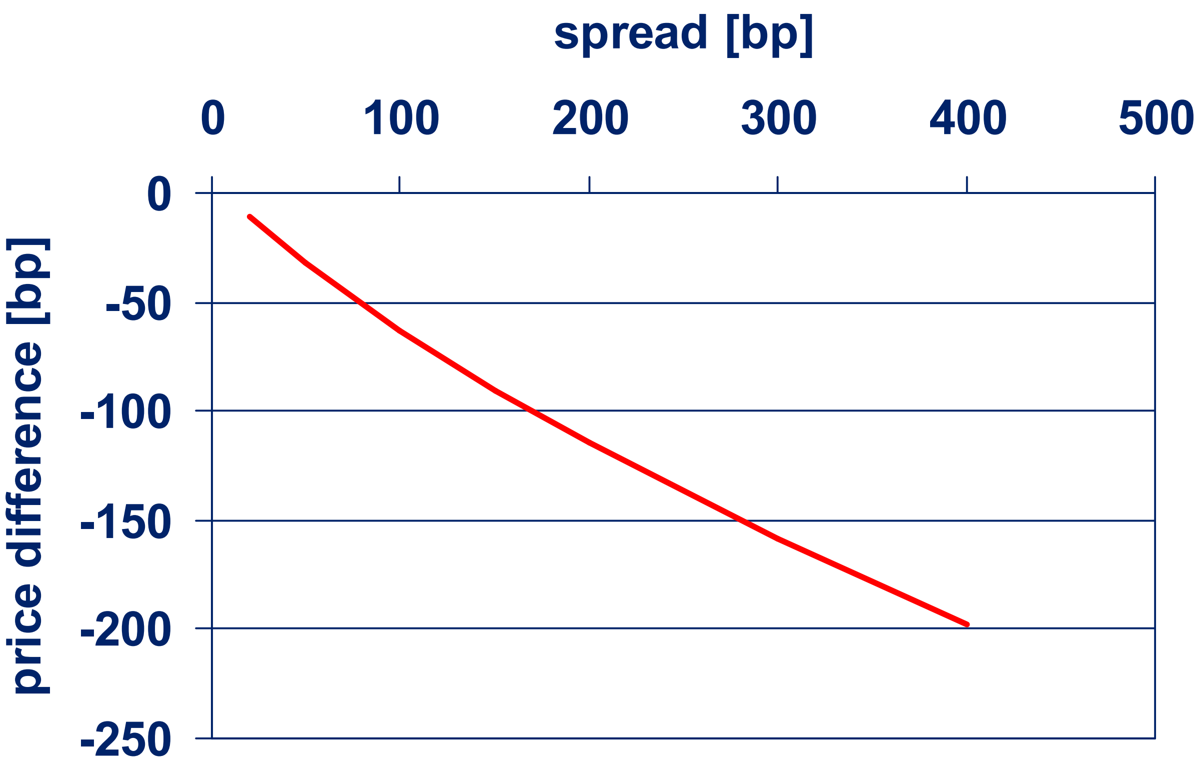


Bermudan callable bond price differences

- Price differences of Bermudan callable bonds with and without stochastic hazard rate

Bond Maturity 10y
Recovery 40%
Correlation -0.3

Spread volatility
50% of spread



Extension to other models

- Separation of positive and negative parts of PV
  - Split claim into two parts

$$V(t,r,\lambda) = \max\left(V(t,r,\lambda),0\right) - \max\left(-V(t,r,\lambda),0\right)$$

- For each claim, determine pay-off in case of default and non-default
  - Calculate recovery rate as the default pay-off fraction of the non-default pay-off
- Netting agreements and collateral management
  - Cannot be implemented on single trade level
  - ⇒ Requires joint modelling of defaultable claims against single counterparty

It's time for your questions!

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