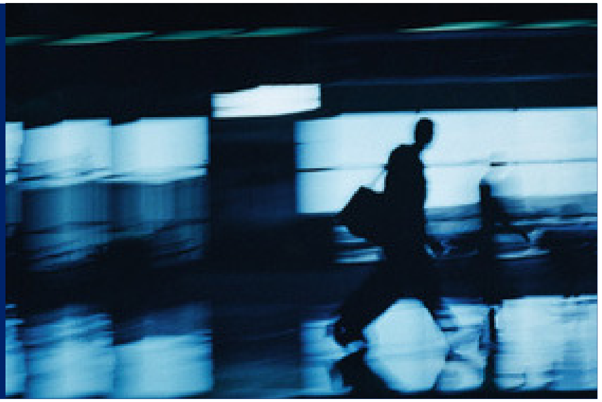
Integration of CDOs into a Merton style portfolio model



Risk Breakfast, London
June 6th, 2008
Dr Anne Kleppe

Agenda

- Motivation
- Economic capital model for credit risk
  - Asset-return process
  - Two-state versus multi-state model
  - Capital allocation
- Incorporating structured products
  - Common features of CDO transactions
  - Modelling assumptions
- Capital allocation for CDOs
  - Modelling correlated default times
  - Approximation by means of large aggregates

Motivation

In the wake of the subprime crisis risk-sensitive modelling of CDO investments becomes more and more important. Financial institutions intend to improve their risk management systems regarding

- Transparency w.r.t. risk parameters of the underlying asset pool
- Risk-sensitivity w.r.t. concentration risk and cross product correlations
- Stress-testing and reporting

→ Accounting for the full risk profile of underlying asset pool is necessary in order to achieve meaningful and risk-sensitive economic capital figures that reflect concentration risk and enable stress reporting.

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Modelling CDO tranches within two-state mode

Model assumptions:

- Model normalized asset log-return process of underlying assets

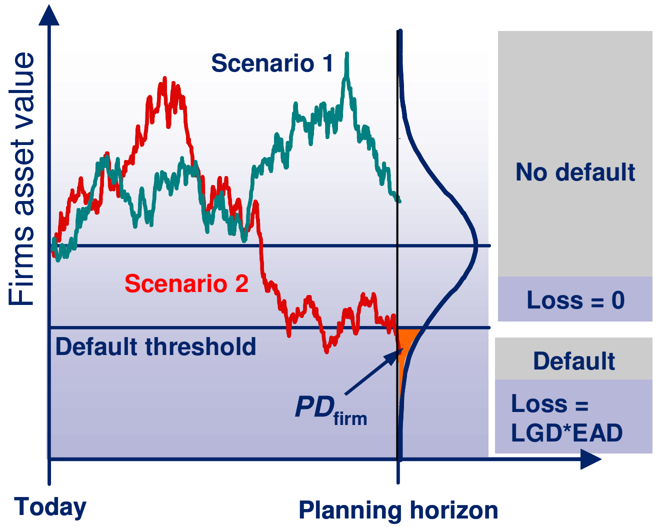
$$\tilde{r}_i = \sum_{g=1}^{N} w_g^{(i)} \cdot \tilde{X}_g + \sqrt{1-R_i^2} \cdot \varepsilon_i$$

- Counterparty defaults if value of its assets $\tilde{r}_i$ falls below value of its liabilities D_i
- Value of liabilities calibrated such that at planning horizon T

$$PD_i = \Pr[\tilde{r}_{i,T} < D_i] = \mathrm{N}[D_i]$$

- Loss of CDO portfolio in two-state mode given by

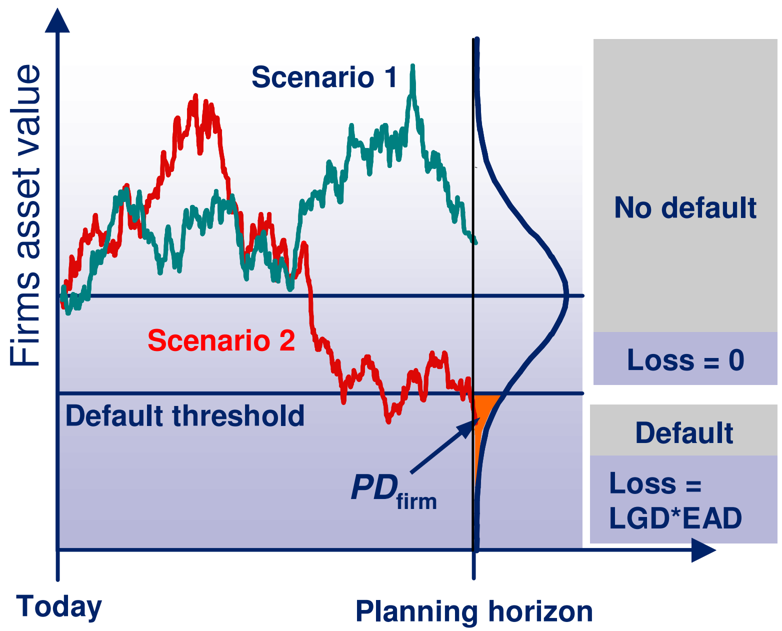
$$\mathrm{Loss}^{\mathrm{CDO\,portfolio}} = \sum_{i=1}^{\#assets} EAD_i \cdot LGD_i \cdot I_{\{\tilde{r}_i < D_i\}}$$



Scenario	Example: Contingent payments			
	Portfolio loss	Equity tranche investors (0 to 100)	Mezzanine tranche investors (100-200)	Senior tranche investors (200-1000)
1	0	0	0	0
2	65	65	0	0
3	20	20	0	0
4	115	100	15	0
5	100	100	0	0
6	300	100	100	100
...				
...				
...				
EL	100	64	19	17

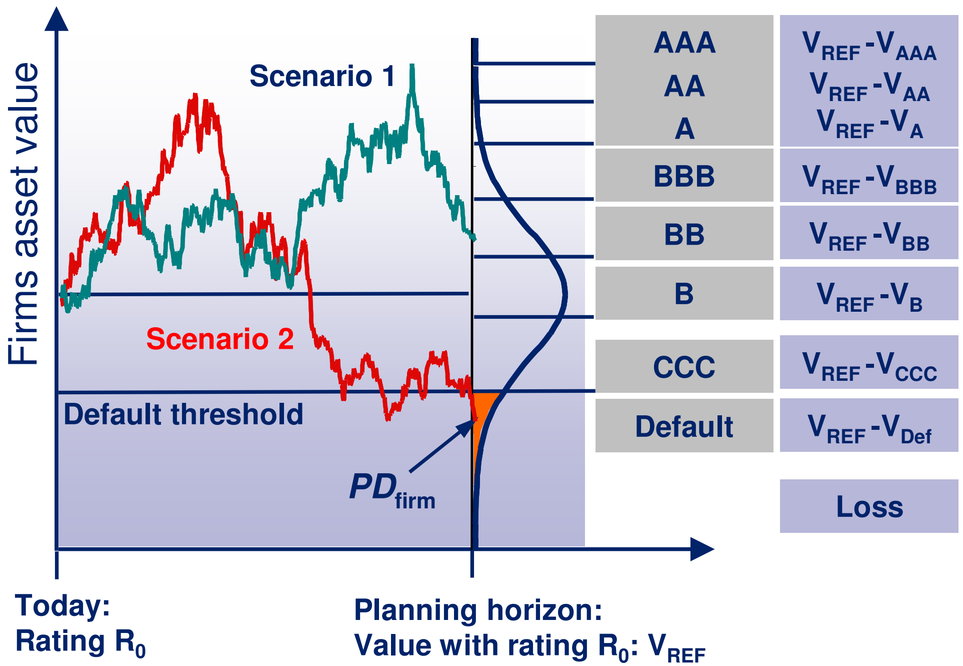
Default mode versus multi-state mode

Default mode



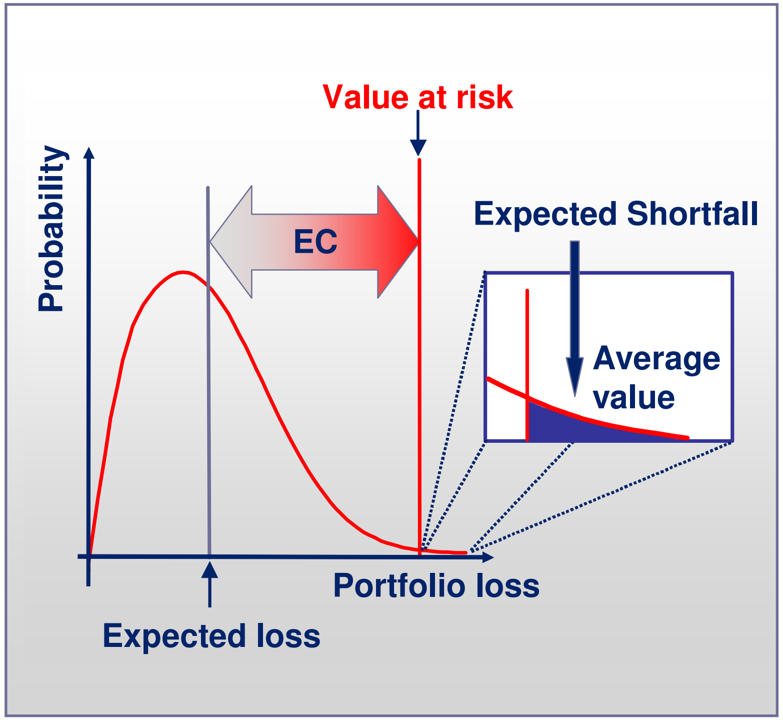
- Event = default/no default
- Loss value depends only on EAD and LGD

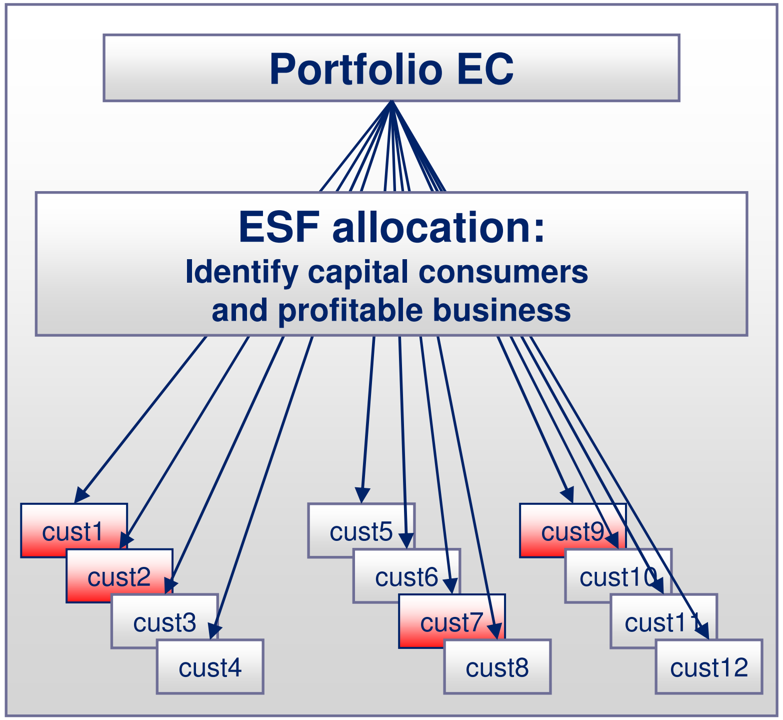
Multi-state mode



- Event = rating change at horizon
- Cash flow valuation: additional dependence on maturity, interest rates, etc.

Capital allocation based on Expected Shortfall*





- Expected shortfall allocation: contributory EC is the average loss of a sub-portfolio i in the "extreme loss scenarios" of the portfolio

$$ESF_i = E\left[Loss_i \middle| Loss_{Portfolio} > \alpha_{Quantile}\right] - E\left[Loss_i\right]$$

*B. Appasamy, AFK, C. Oehler, "Robust and Stable Capital Allocation", Wilmott article, May 2008

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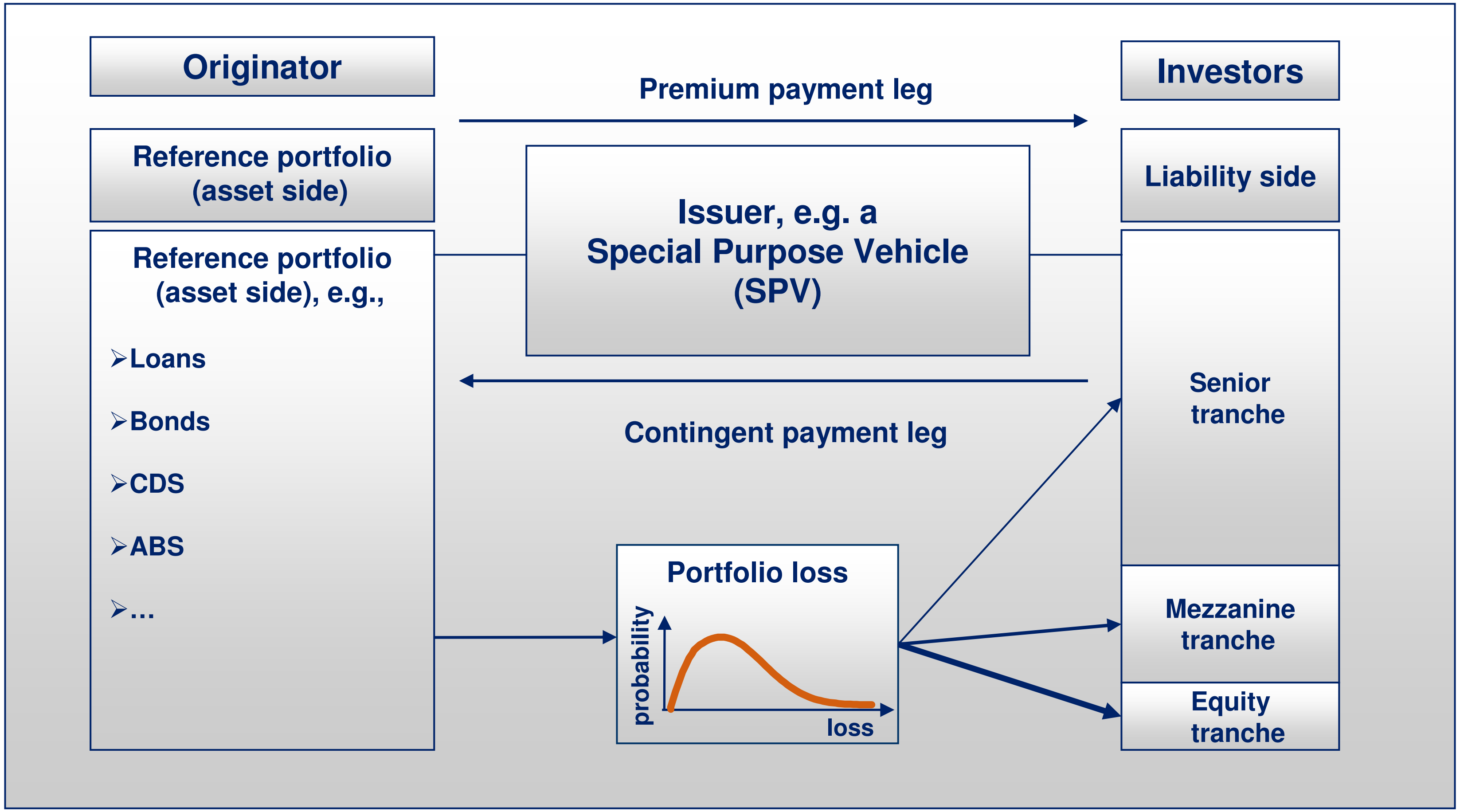
Common features of CDO transactions



Modelling assumptions

Due to the complexity of CDOs simplifying modelling assumptions are made:

- Transaction costs are ignored
- Active management is not taken into account
- Triggers, such as market-value or rating triggers etc., are not taken into account
- Contingent payments for defaults within some time interval $[t\text{-}1,t]$ are made at the same time as premium payments for this time period

⇨ Most importantly the objective here is not to match market prices but to estimate the economic capital contribution of CDO tranches to the portfolio economic capital.

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Calculation of scenario losses at risk horizon

- Within each Monte-Carlo scenario need to calculate scenario specific loss separately for each CDO tranche

$$Loss_{scenario_i}^{tranche_n} = PV_{scenario_i}^{tranche_n} - \langle PV \rangle_{tranche_n}$$

- Expected value of the CDO tranche

$$\langle PV \rangle_{tranche_n} = \frac{1}{N} \sum_{i=1}^{N} PV_{scenario_i}^{tranche_n}$$

- The scenario value of a CDO tranche is given by the difference between the contingent payment leg and the premium payment leg

$$PV_{scenario_i}^{tranche_n} = PV_{contingent\ \ payment\ \ leg}^{scenario_i} - PV_{premium\ \ payment\ \ leg}^{scenario_i}$$

- Contingent payment leg

$$PV_{contingent\ payment\ leg}^{scenario_i} = \sum_{t=1}^{T} d_t^0 \left(L_t^{scenario_i} - L_{t-1}^{scenario_i} \right)$$

- Premium payment leg

$$PV_{premium\ \ payment\ leg}^{scenario_i} = \sum_{t=1}^{T} d_t^0 c^{tranche} \left(Notional^{tranche} - L_t^{scenario_i} \right)$$

Calculation of tranche losses within given time interval

- The main input parameter for both the scenario specific premium and contingent payment leg is the loss $L_t^{scenario_i}$ of the tranche up to time t

$$L_t^{scenario_i} = \min\left(UB - LB, \max\left(x_t^{scenario_i} - LB, 0\right)\right)$$

$x_t^{scenario_i} =$ Accumulated losses of reference portfolio up to time t

$\max(x_t^{scenario_i}) =$ Notional of CDO portfolio

$LB =$ Lower bound of tranche (attachment point)

$UB =$ Upper bound of tranche (detachment point)

Time interval	# defaults	Loss within time interval	Accumulated loss
$t_1 - t_0$	n_1	$Loss_1 = \sum_{j=1}^{n_1} EAD_j LGD_j$	$x_1^{scenario_i} = Loss_1$
$t_2 - t_1$	n_2	$Loss_2 = \sum_{i=1}^{n_2} EAD_j LGD_j$	$x_2^{scenario_i} = Loss_1 + Loss_2$
…	…	…	…
$t_T - t_{T-1}$	n_T	$Loss_T = \sum_{j=1}^{n_T} EAD_j LGD_j$	$x_T^{scenario_i} = \sum_{j=1}^{T} Loss_j$

Ingredients for an efficient risk-sensitive approximation

Within each scenario we need to simulate the accumulated losses $x_t^{scenario_i}$ of the underlying reference portfolio up to time t (with $t_{\max} = T$, the maturity of the CDO)

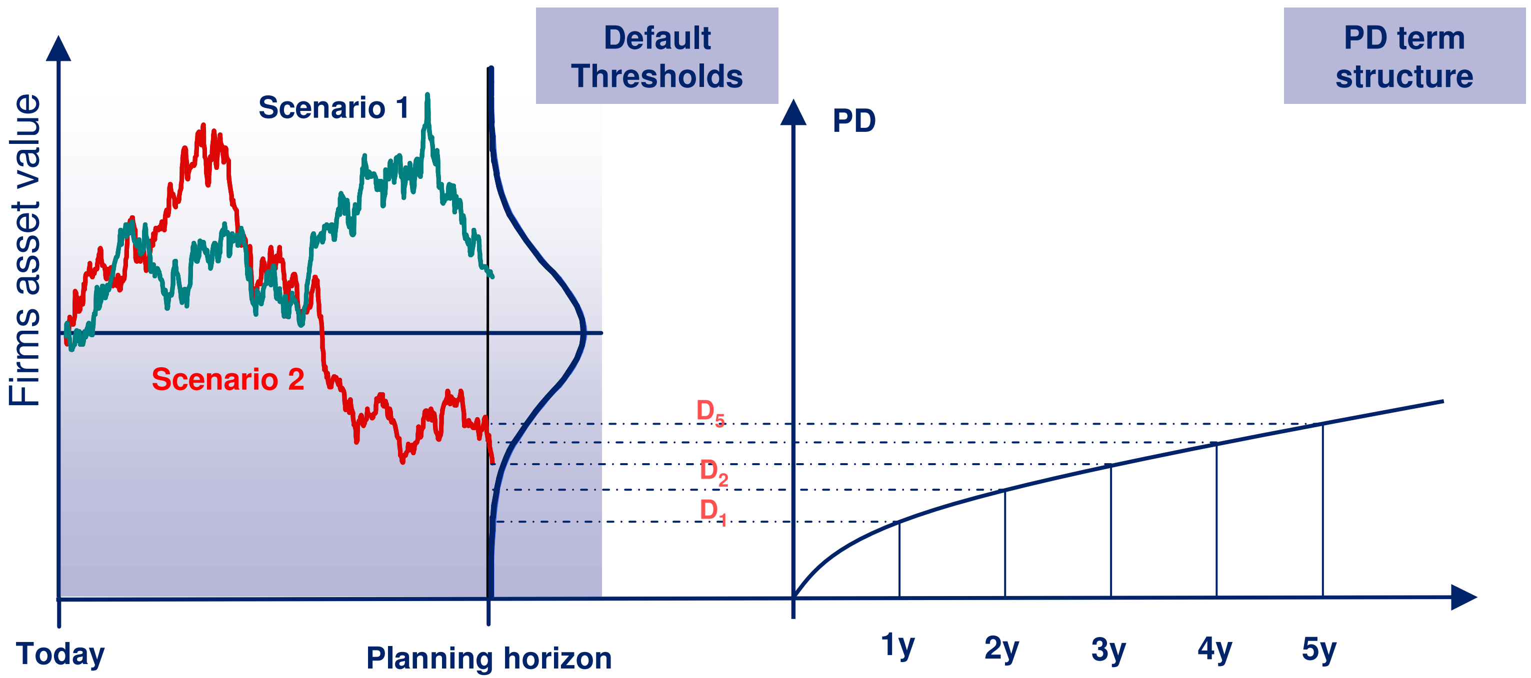
- Problem 1:
  In order to simulate the accumulated losses within some time interval we need to specify the default times of the underlying assets of the reference portfolio.

  ⇒ Translate the PD-term structure into individual default times of the underlying assets of the reference portfolio.

- Problem 2:
  In principal need to simulate the default times for each underlying asset separately. This is due to the large number of underlyings computationally by far to cost-intensive.

  ⇒ "Bundle" the underlying assets of the reference portfolio into large aggregates of the same risk characteristics and use the "Law of Large Numbers" (LLN) in order to estimate the number of defaults within these aggregates.

Inferring the default barriers from the PD term structure

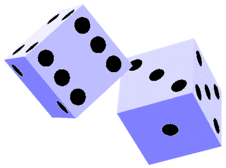


From the PD term structure one can therefore infer the default barriers D_t^i for a facility i to default within time t by setting

$$D_t^i = N^{-1}(p_t^i)$$

Thus from $D_{t-1}^i < \tilde{r}_i \leq D_t^i$ one can infer that facility i will default within time t-1 to t.

Determine the number of defaults within an aggregate

Bundle underlyings into large aggregates	Simulate systematic risk	Apply Law of Large Numbers
Assets within an aggregate have common risk parameters R^2, PD, LGD, EAD and are described by the same composite factor but differ in their idiosyncratic risk	 Given the systematic risk the individual defaults within an aggregate are independent	The fraction of assets that defaults within a large aggregate is proportional to the conditional default probability $n_t \sim N\left(\frac{D_t - \sum w_g \tilde{X}_g}{\sqrt{1-R^2}}\right)$

Calculating the EC contribution of the tranche

Within each scenario we have generated a scenario loss

$$Loss_{scenario_i}^{tranche_n} = PV_{scenario_i}^{tranche_n} - \langle PV \rangle_{tranche_n}$$

In order to calculate the EC contribution of the CDO tranche to the portfolio EC based on expected shortfall we have to average over all scenario losses given that the total portfolio loss is greater than some pre-specified quantile *α:*

$$ESF_{scenario_i}^{tranche_n} = \frac{1}{N}\sum_{i=1}^{N}\left(Loss_{scenario_i}^{tranche_n} \middle| Loss_{scenario_i}^{Portfolio} > \alpha_{Quantile}\right) - \underbrace{E\left[Loss_{scenario_i}^{tranche_n}\right]}_{=0}$$

Conclusion

Methodology allows for

- High transparency with respect to riskiness of underlying assets
- Sensitivity with respect to concentration risk:
  e.g. single name concentrations, cross product correlations
- Stress testing with respect to deteriorated credit risk parameters of the underlying asset pools:
  e.g. default correlations, PD, LGD, migration probabilities, etc…

In the context of the subprime crisis a risk-sensitive modelling of CDO investments would have provided early warning signals reportable to Senior Management and would subsequently have enabled appropriate action.

In order to avoid the same financial disaster, accounting for the true risk-profile of CDO investments is –besides many other issues, of course – indispensable!

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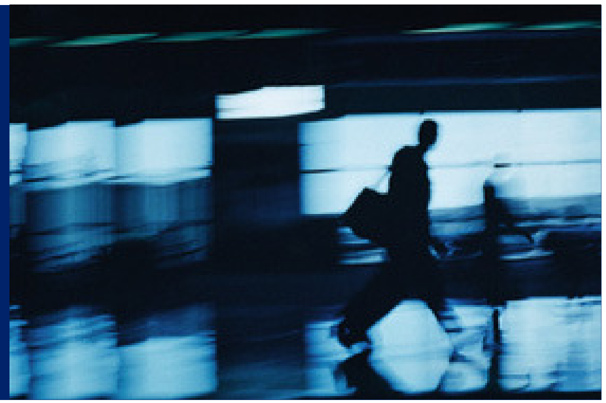
Backup slides



Segment Facilities

As it is by far to time consuming to simulate all underlyings separately it is necessary to aggregate several underlyings yielding a segment facility having the following properties:

- Risk parameters for all facilities within the segment are the same:
  - PD term structure is the same for all facilities
  - R2 is the same for all facilities
  - Weights (and systematic risk factors) are the same for all facilities
  - Exposure is the same for all facilities
- A segment facility is thus an aggregation of s single facilities having the same properties but different idiosyncratic risks.

Loss of a Segment Facility (1)

The fraction of obligors within a segment facility having defaulted within time t is determined as a function of the systematic factors x_j , $j = 1, \ldots, m$.

Let D^i_t be the default threshold of the segment i under consideration to default within time t, i.e. if for an obligor z, $z = 1, \ldots, s$, in this segment facility $\tilde{r}_z < D^i_t$ than obligor z defaults within time t.

Assume the number of obligors s_i in the segment facility is large enough to apply the law of large numbers.

Then the number m^i_t of defaults in the segment facility within time t is given by:

$$m_t^i = \sum_{z=1}^{s_i} I_{\{\tilde{r}_z < D_t^i\}} = \sum_{z=1}^{s_i} I_{\{R_i \tilde{X}_i + \sqrt{1-R_i^2} Z_z < D_t^i\}} = s_i \left[\frac{1}{s_i} \sum_{z=1}^{s_i} I_{\{Z_z < \phi_t^i\}} \right] \quad \text{with} \quad \phi_t^i = \frac{D_t^i - R_i \tilde{X}_i}{\sqrt{1-R_i^2}}$$

From this we can infer the number $m^i_{t\text{-}\varepsilon:t}$ of defaults within time t-ε to t as

$$m_{t-\varepsilon:t}^i = m_t^i - m_{t-\varepsilon}^i$$

Loss of a Segment Facility (2)

The number m^i_t of defaults in the segment facility i within time t is given by

$$m_t^i = \sum_{z=1}^{s_i} I_{\{\tilde{r}_z < D_t^i\}} = \sum_{z=1}^{s_i} I_{\{R_i \tilde{X}_i + \sqrt{1-R_i^2} Z_z < D_t^i\}} = s_i \left[\frac{1}{s_i} \sum_{z=1}^{s_i} I_{\{Z_z < \phi_t^i\}} \right] \quad \text{with} \quad \phi_t^i = \frac{D_t^i - R_i \tilde{X}_i}{\sqrt{1-R_i^2}}$$

Since $\tilde{X}_i$ is given in an MC scenario we know ϕ_t^i . Further, $I_{\{Z_z < \phi_t^i\}}$ are independent random variables and the approximation according to the law of large numbers can be applied

$$\frac{1}{s_i} \sum_{z=1}^{s_i} I_{\{Z_z < \phi_t^i\}} \xrightarrow{s_i \to \infty} P\left[Z < \phi_t^i\right] = N(\phi_t^i)$$

Altogether we conclude that given the systematic risk $R_i \tilde{X}_i$ within the segment i the fraction of obligors having defaulted within time t is given by

$$\frac{m_t^i}{s_i} = N\left(\phi_t^i\right) = N\left(\frac{D_t^i - R_i \tilde{X}_i}{\sqrt{1-R_i^2}} \right)$$

and finally the loss of a segment facility within time t-ε to t is given by

$$Loss_i\left(\tilde{X}_i\right) = m_{t-\varepsilon:t}^i \cdot LGD_i \cdot EAD_i = s_i \cdot \left(N\left(\phi_t^i\right) - N\left(\phi_{t-\varepsilon}^i\right)\right) \cdot LGD_i \cdot EAD_i = s_i \cdot \left(N\left(\frac{D_t^i - R_i \tilde{X}_i}{\sqrt{1-R_i^2}} \right) - N\left(\frac{D_{t-\varepsilon}^i - R_i \tilde{X}_i}{\sqrt{1-R_i^2}} \right) \right) \cdot LGD_i \cdot EAD_i$$