Value at Risk

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Fundamentals

- The risk of a financial instrument or a portfolio is induced by (fluctuations of) its underlyings, called *risk factors*.
    - examples of risk factors: 3-month LIBOR rate, US$/Euro exchange rate, DAX index.
- The portfolio value is a function of stochastic processes (the risk factors) and is

thus itself a stochastic process.

- A *scenario* for a portfolio is defined by specifying a value for each of its risk factors.
- The *Value at Risk* is an upper bound for the loss incurred by a portfolio
  - which – with a probability c – will not be exceeded during some (finite) time period δt.
  - The probability c is referred to as the *confidence* or *level of confidence.*
- A *confidence interval* of a stochastic variable x is the subset of the variable's range (the interval) attained with a confidence c.
  - *symmetric confidence interval* (where $p(x)$ denotes the probability density) around the mean μ

$$c \stackrel{!}{=} P(\mu - a < x < \mu + a) = \int_{\mu-a}^{\mu+a} p(x)dx$$

- *one-sided confidence interval*

$$c \overset{!}{=} P(x > a) = 1 - P(x \leq a) = 1 - \int_{-\infty}^{a} p(x)dx \tag{1.1}$$

- The *percentile* or *quantile* associated with a probability c is the value Q_c such that the probability that a random value x is less than or equal to Q_c is equal to c:

$$P(x \leq Q_c) = c \Longleftrightarrow \int_{-\infty}^{Q_c} p(x)dx = c$$

  - Thus, the percentile is the "inverted" cumulative distribution function P:

$$Q_c = P^{-1}(c)$$

  - The boundary a of a one-sided confidence interval is thus the $(1-c)$-

percentile

$$\begin{aligned} c \stackrel{!}{=} P(x > a) &= 1 - P(x \leq a) \\ P(x \leq a) &= 1 - c \\ a = Q_{1-c} &= P^{-1}(1 - c) \end{aligned}$$

- The *Value at Risk* (VaR) of the value V of a financial instrument or a portfolio is the upper bound for the *loss* which will not be exceeded with confidence c over a time span δt .

$$c \stackrel{!}{=} \text{cpf}_{\delta V}(\delta V > -\text{VaR}(c)) \tag{1.2}$$

  - δV denotes the change of V and $\text{cpf}_{\delta V}$ (instead of P) denotes the *cumulative probability function* of the random variable δV.
  - more explicitly

$$c \stackrel{!}{=} 1 - \text{cpf}_{\delta V}(\delta V \leq -\text{VaR}(c)) = 1 - \int_{-\infty}^{-\text{VaR}(c)} \text{pdf}_{\delta V}(x) dx \tag{1.3}$$

  - $\text{pdf}_{\delta V}(x)$ denotes the *probability density function* of the random variable δV.

– Therefore the *negative* VaR is the $(1-c)$-percentile of the distribution of δV :

$$\mathrm{VaR}(c) = -Q_{1-c}^{\mathrm{cpf}_{\delta V}} = -\mathrm{cpf}_{\delta V}^{-1}(1-c) \tag{1.4}$$

- The Value at Risk is defined by the probability distribution of δV, *not* by the probability distributions of the risk factors!

- Using the probability distribution of the *risk factor* instead of the probability distribution of the *portfolio* is only possible when V is a *monotonous* function of the risk factor process S.

  – Only then we have

$$\begin{aligned} \mathrm{VaR}_V(c) &= V(S) - \min\{V(S=a), V(S=\widetilde{a})\} \\ &= \max\{V(S) - V(S=a), V(S) - V(S=\widetilde{a})\} \end{aligned} \tag{1.5}$$

  – where a and $\widetilde{a}$ denote the lower and upper boundary of the risk factor's confidence interval. The upper bound $\widetilde{a}$ is defined analogously to Equation

1.1 through

$$c \overset{!}{=} P(x < \widetilde{a}) = \int_{-\infty}^{\widetilde{a}} p(x)dx$$

1.1 Risk Factor Evolution

- Single risk factor S governed by a *geometric Brownian motion* (*GBM*)

$$d \ln S(t) = \mu dt + \sigma dW \text{ with } dW \sim \mathrm{N}(0, dt) \tag{1.6}$$

- What is the process for S itself?
  - Ito's Lemma:

$$df(S,t) = \left[\mu \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial S^2}\right] dt + \frac{\partial f}{\partial S} \sigma \, dW \tag{1.7}$$

– Define the stochastic variable $y := \ln(S(t))$. Thus y satisfies

$$dy(t) = \mu dt + \sigma\, dW$$

– Choose the function f as $f(y,t) = e^y$.

$$f(y,t) = e^y \quad \Rightarrow \quad \frac{\partial f}{\partial y} = f \quad , \quad \frac{\partial f}{\partial t} = 0\,, \quad \frac{\partial^2 f}{\partial y^2} = f$$

$$df(y,t) = \left(\mu f(y,t) + \frac{\sigma^2}{2} f(y,t)\right) dt + f(y,t)\, \sigma\, dW$$

– Since $f(y,t) = S(t)$ we have for the *differential* of S:

$$dS(t) = S(t) \underbrace{\left(\mu + \frac{\sigma^2}{2}\right)}_{\widetilde{\mu}} dt + S(t) \sigma\, dW \tag{1.8}$$

- What is the process for the risk factor over a *finite* time?

  – *finite* changes in S (over a finite, positive time span δt) can be derived by solving the SPDE 1.8

- Define a stochastic variable $y = W(t)$= the value of the Wiener process at time t. This satisfies

$$dy(t) = 0dt + 1dW(t)$$

- Construct a function S of the stochastic variable y by

$$S(y,t) := S_0 \exp\left(\mu t + \sigma y\right)$$

  where S_0 is an arbitrary factor.

- Ito's lemma gives the process for S induced by the process dy:

$$\begin{aligned} dS &= \left[\underbrace{\frac{\partial S}{\partial y}}_{\sigma S} 0 + \underbrace{\frac{\partial S}{\partial t}}_{\mu S} + \frac{1^2}{2} \underbrace{\frac{\partial^2 S}{\partial y^2}}_{\sigma^2 S} \right] dt + \underbrace{\frac{\partial S}{\partial y}}_{\sigma S} 1dW \\ &= (\mu + \frac{\sigma^2}{2}) S\, dt + \sigma S\, dW \end{aligned}$$

- This corresponds exactly to the process in Equation 1.8. The process S thus constructed is therefore a solution of the SPDE 1.8.

- Simply making the substitution $t \to t + \delta t$ we obtain

$$\begin{aligned} S(t+\delta t) &= S_0 \exp\left(\mu t + \mu \delta t + \sigma y(t+\delta t)\right) \\ &= \underbrace{S_0 \exp\left(\mu t\right)}_{\widetilde{S}_0(t)} \exp\left(\mu \delta t + \sigma W(t+\delta t)\right) \\ &= \widetilde{S}_0(t) \exp\left(\sigma W(t) + \mu \delta t + \sigma \delta W\right) \end{aligned}$$

  with the notation δW for a change in a Brownian motion after the passing of a finite time interval δt:

$$\delta W := W(t+\delta t) - W(t) \Longrightarrow \delta W \sim N(0, \delta t) \tag{1.9}$$

- The first term in the exponent refers to (already known) values at time t. It can also be absorbed into the (still arbitrary) pre-factor

$$S(t+\delta t) = \underbrace{\widetilde{S}_0(t) e^{\sigma W(t)}}_{\widetilde{\widetilde{S}}_0(t)} \exp\left(\mu \delta t + \sigma \delta W\right)$$

- Initial condition for the solution of the SPDE:

$$S(t+\delta t) \stackrel{\delta t \to 0}{=} S(t) \Longrightarrow \widetilde{\widetilde{S}}_0(t) = S(t)$$

- Thus, we obtain the change in S corresponding to Equation 1.8 or 1.6 over a finite, positive time span δt:

$$S(t+\delta t) = S(t)\exp\left(\mu\delta t + \sigma\delta W\right) \text{ with } \delta W \sim \mathrm{N}(0,\delta t) \tag{1.10}$$

- The finite time span δt can be taken to be arbitrarily long. δt is taken to be the *liquidation period*.

1.2 Value at Risk of a Single Risk Factor

- Consider a portfolio consisting of a single position in N of the same risk factor S.

$$V = NS(t)\ ,\ \ \delta V(t) = N\delta S(t) \ \text{ with } \ \delta S(t) = S(t+\delta t) - S(t)$$

  - The change in S induces a change in V amplified by the *constant* factor N.
  - The factor N is the *sensitivity* of V with respect to S.
  - δV is a *linear* function of δS.

- The value change δV over the liquidation period δt follows directly from Equation 1.10

$$\begin{aligned}\delta V &= NS(t+\delta t) - NS(t) \\ &= NS(t)\left[\exp\left(\mu\delta t + \sigma\delta W\right) - 1\right]\end{aligned} \tag{1.11}$$

- The VaR as defined in Equation 1.3 is:

$$\mathrm{cpf}_{\delta V}(\delta V \leq -\mathrm{VaR}) = \mathrm{cpf}_{\delta V}\left(NS(t)\left[e^{\mu\delta t+\sigma\delta W} - 1\right] \leq -\mathrm{VaR}\right)$$

  - $\mathrm{cpf}_{\delta V}$ is unknown. But: The only stochastic variable involved is the Brownian motion:

$$\delta W \sim \mathrm{N}(0, \delta t) \Rightarrow \delta W \sim X\sqrt{\delta t} \quad \text{with} \quad X \sim \mathrm{N}(0,1)$$

  - Rewrite the event that $\delta V \leq -\mathrm{VaR}$ with the purpose of isolating δW.

$$\begin{aligned}\mathrm{cpf}_{\delta V}(\delta V \leq -\mathrm{VaR}) &= \mathrm{cpf}_{\delta V}\left(\delta W \leq \frac{\ln\left(1 - \frac{\mathrm{VaR}}{NS(t)}\right) - \mu\delta t}{\sigma}\right) \\ &= \mathrm{cpf}_{\delta V}\left(\delta W \leq a\sqrt{\delta t}\right)\end{aligned}$$

with the abbreviation

$$a := \frac{\ln\left(1 - \frac{\text{VaR}}{NS(t)}\right) - \mu\delta t}{\sigma\sqrt{\delta t}} \tag{1.12}$$

- The probability that δW is less than or equal to a certain value is dependent on the distribution of δW alone and not on that of δV. We can therefore simply replace $\text{cpf}_{\delta V}$ with $\text{cpf}_{\delta W}$:

$$\begin{aligned} \text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) &= \text{cpf}_{\delta W}\left(\delta W \leq a\sqrt{\delta t}\right) \\ &= \text{cpf}_{\delta W}\left(X\sqrt{\delta t} \leq a\sqrt{\delta t}\right) \\ &= \text{cpf}_{\delta W}\left(X \leq a\right) \end{aligned}$$

- The probability that X is smaller than a particular variable is dependent on the distribution of X alone and not on that of δW allowing $\text{cpf}_{\delta W}$ to be simply replaced by cpf_X =N$(0,1)$

$$\text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) = \text{cpf}_X\left(X \leq a\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} \exp(-x^2/2)dx$$

- The Value at Risk with respect to a confidence c follows now from the requirement

$$c \stackrel{!}{=} 1 - \text{cpf}_{\delta V}(\delta V \leq -\text{VaR}) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp(-x^2/2)dx$$

- a in 1.12 is thus the $(1-c)$-percentile of the standard normal distribution

$$a = Q_{1-c}^{\mathrm{N}(0,1)}$$

- Solving 1.12 for VaR finally yields the value at risk of a long position in N risk factors:

$$\text{VaR}(c) = NS(t)\left[1 - \exp\left(\mu\delta t + Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right]$$

  - Examples of percentiles of the standard normal distribution:

$$\begin{aligned} c = 95\% = 0,95 &\Rightarrow a = Q_{1-c}^{\mathrm{N}(0,1)} \approx -1,65 \\ c = 99\% = 0,99 &\Rightarrow a = Q_{1-c}^{\mathrm{N}(0,1)} \approx -2,326 \end{aligned} \tag{1.13}$$

- Now: Risk of a *short* position consisting of N of the risk factor.

$$\delta V = -NS(t)\left[\exp\left(\mu\delta t + \sigma\delta W\right) - 1\right]$$

– Because of the minus sign: Start from the original definition, Equation 1.2.

$$\begin{aligned}
\mathrm{cpf}_{\delta V}(\delta V > -\mathrm{VaR}) &= \mathrm{cpf}_{\delta V}(NS(t)\left[e^{\mu\delta t+\sigma\delta W} - 1\right] < \mathrm{VaR}) \\
&= \mathrm{cpf}_{\delta W}\left(\delta W < \frac{\ln\left(1 + \frac{\mathrm{VaR}}{NS(t)}\right) - \mu\delta t}{\sigma}\right) \\
&= \mathrm{cpf}_{X}\left(X < \frac{\ln\left(1 + \frac{\mathrm{VaR}}{NS(t)}\right) - \mu\delta t}{\sigma\sqrt{\delta t}}\right)
\end{aligned}$$

- Define an abbreviation analogously to Equation 1.12:

$$\widetilde{a} = \frac{\ln\left(1 + \frac{\mathrm{VaR}}{NS(t)}\right) - \mu\delta t}{\sigma\sqrt{\delta t}} \tag{1.14}$$

- The Var follows from the requirement

$$c \stackrel{!}{=} \mathrm{cpf}_{\delta V}(\delta V > -\mathrm{VaR}) = \mathrm{cpf}_{X}\left(X < \widetilde{a}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\widetilde{a}} \exp(-x^2/2)dx$$

– $\widetilde{a}$ is the c-percentile of the standard normal distribution

$$\widetilde{a} = Q_c^{\mathrm{N}(0,1)} = -Q_{1-c}^{\mathrm{N}(0,1)}$$

.

– The last step follows from the symmetry of the standard normal distribution:

$$\mathrm{N}(-x) = 1 - \mathrm{N}(x) \Longrightarrow Q_c^{\mathrm{N}(0,1)} = -Q_{1-c}^{\mathrm{N}(0,1)}$$

– Now solve1.14 for the VaR of a short position in $-N$ risk factors S:

$$\mathrm{VaR}(c) = -NS(t)\left[1 - \exp\left(\mu\delta t - Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right]$$

- Summary:

$$\begin{aligned}\mathrm{VaR}_{\mathrm{long}}(c) &= NS(t)\left[1 - \exp\left(\mu\delta t + Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right] \\ \mathrm{VaR}_{\mathrm{short}}(c) &= -NS(t)\left[1 - \exp\left(\mu\delta t - Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right] \qquad (1.15)\end{aligned}$$

- These two VaR's are *not* equal in magnitude, due to

– the drift μ
– *log*normally distributed price changes. The lognormal distribution is not symmetric.

1.3 Approximation for the Risk Factors

- For short liquidation periods (e.g., $\delta t = 10$ days $= 0,0274$ years): $\exp(x) \approx 1+x$ and/or $\mu \approx 0$:

$$\delta S(t) \approx \begin{cases} S(t)\left[e^{\sigma\delta W} - 1\right] & \mu \text{ neglected} \\ S(t)\left[\mu\delta t + \sigma\delta W\right] & \text{linear proxy for exp} \\ S(t)\sigma\delta W & \mu \text{ neglected and linear proxy} \end{cases} \tag{1.16}$$

  – That means for the Value at Risk of a *long* position in N risk factors S

$$\mathrm{VaR}_{\mathrm{long}}(c) \approx \begin{cases} NS(t)\left[1 - \exp\left(Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right] & \mu \approx 0 \\ NS(t)\left[-\mu\delta t - Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right] & \text{exp linear} \\ -NS(t)Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t} & \mu \approx 0,\text{exp linear} \end{cases} \tag{1.17}$$

- VaR of the *short* position: the only differences are signs of N and $Q_{1-c}^{\mathrm{N}(0,1)}$

- Long and Short VaR equal *only if*
  - linear approximation *and*
  - drift is neglected *and*
  - portfolios value changes are (approximately) linearl function of underlying risk factor (Delta-Normal approximation)

- Only in this case: square root of time law and the linear scaling with percentiles:

$$\mathrm{VaR}(c', \delta t') \approx \frac{Q_{1-c'}}{Q_{1-c}} \sqrt{\frac{\delta t'}{\delta t}} \; \mathrm{VaR}(c, \delta t) \tag{1.18}$$

1.4 The Covariance Matrix

- In general n (often hundreds of) risk factors modelled by random walks obeying the *coupled* stochastic differential equations

$$d \ln S_i(t) = \mu_i dt + dZ_i \quad \text{for} \quad i = 1, 2, \ldots, n \tag{1.19}$$

- Here dZ_i are *correlated* drift-free Brownian motions with *covariance*

$$\begin{aligned} \mathrm{cov}[dZ_i, dZ_j] &= d\Sigma_{ij} \ , \quad \mathrm{E}\,[dZ_i] = 0 \\ \text{with} \quad d\Sigma_{ij} &= \rho_{ij}\ \sigma_i \sqrt{\delta t}\ \sigma_j \sqrt{\delta t} = \sigma_i \rho_{ij} \sigma_j dt \end{aligned} \tag{1.20}$$

  - σ_i = *volatility* of Z_i, i.e. of risk factor S_i
  - ρ_{ij} = *correlation* between Z_i and Z_j , i.e. between S_i and S_j
  - $d\Sigma_{ij}$ is called *covariance matrix.*

- Notation:

$$\begin{aligned} &\mathbf{X} \sim \mathrm{N}(\mathbf{R}, \mathbf{V}) \Longleftrightarrow \\ &X_i \text{ normally distributed with} \\ &\mathrm{cov}[X_i, X_j] = V_{ij} \ , \quad \mathrm{E}\,[X_i] = R_i \end{aligned}$$

  i.e.

$$d\mathbf{Z} = \begin{pmatrix} dZ_1 \\ dZ_2 \\ \vdots \\ dZ_n \end{pmatrix} \sim \mathrm{N}(\mathbf{0}, d\mathbf{\Sigma})$$

- After a finite time δt:
  - Solutions as in 1.10 to the stochastic differential equations 1.19:

$$S_i(t+\delta t) = S_i(t)\exp\left(\mu_i \delta t + \delta Z_i\right) \text{ with } \delta \mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \delta \mathbf{\Sigma}) \tag{1.21}$$

  - with the covariance matrix

$$\delta \mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \ddots & & \ddots & \Sigma_{2n} \\ \vdots & & \Sigma_{ij} & & \vdots \\ \vdots & \ddots & & \ddots & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \cdots & \cdots & \Sigma_{nn} \end{pmatrix} \text{ where } \delta \Sigma_{ij} = \sigma_i \rho_{i,j} \sigma_j \delta t$$

$$\text{for } \; i,j = 1,2,\ldots,n \tag{1.22}$$

  - Here, δt is the reference time interval for the change in the risk factors,
    - i.e. $\delta t = 1$ day for a daily changes in the risk factors,
    - $\delta t = 25$ days for a monthly change, etc.

- Proxy for $\delta S_i(t) = S_i(t+\delta t) - S_i(t)$ over a short time δt:

$$\delta S_i(t) \approx \begin{cases} S_i(t)\left[e^{\delta Z_i} - 1\right] & \text{drift neglected} \\ S(t)\left[\mu_i \delta t + \delta Z_i\right] & \text{linear proxy for exp} \\ S(t)\delta Z_i & \text{drift neglected and linear proxy} \end{cases} \tag{1.23}$$

  - Usually:

$$\delta S_i(t) \approx S_i(t)\, \delta Z_i \quad \text{with} \quad \delta \mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \delta \mathbf{\Sigma}) \tag{1.24}$$

1.4.1 Cholesky-Decomposition of the Covariance Matrix

- The symbol $\mathbf{1}$ denotes the *identity matrix*

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \Longleftrightarrow (\mathbf{1})_{ij} = \delta_{ij} \tag{1.25}$$

- The matrix $\mathbf{A}$ denotes the “square root” of the covariance matrix

$$\mathbf{A}\mathbf{A}^T = \delta \mathbf{\Sigma} \tag{1.26}$$

- $\mathbf{A}^T$ denotes the *transpose* of the matrix $\mathbf{A}$:

$$\left(\mathbf{A}^T\right)_{ij} = A_{ji}$$

- $\mathbf{A}$ transforms uncorrelated random variables into correlated ones (with covariance $\delta\mathbf{\Sigma}$):

  - Let $X_i, i = 1, \dots n$ be uncorrelated:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ with } \mathbf{X} \sim \mathrm{N}(\mathbf{0}, \mathbf{1}) \qquad (1.27)$$

    * more xeplicitly

$$X_i \sim \mathrm{N}(0,1)\ \forall i = 1, \dots n,\ \ \mathrm{cov}[X_i, X_j] = \delta_{ij}\ \ \forall i, j = 1, \dots n$$

- Applying the matrix $\mathbf{A}$ to $\mathbf{X}$ generates new random variables $\mathbf{Y}$:

$$\mathbf{Y} = \mathbf{AX} \Longleftrightarrow Y_i = \sum_k A_{ik} X_k$$

- Since this is a linear transformation, the Y_i are also normally distributed.
- Covariances of teh new variables:

$$\begin{aligned} \mathrm{cov}[Y_i, Y_j] &= \mathrm{cov}[\sum_k A_{ik} X_k, \sum_m A_{jm} X_m] \\ &= \sum_k A_{ik} \sum_m A_{jm} \underbrace{\mathrm{cov}[X_k, X_m]}_{\delta_{km}} \\ &= \sum_k A_{ik} A_{jk} = \left(\mathbf{AA}^T\right)_{ij} = \delta\Sigma_{ij} \end{aligned}$$

- The expectations of these random variables are

$$\mathrm{E}[Y_i] = \mathrm{E}[\sum_k A_{ik} X_k] = \sum_k A_{ik} \underbrace{\mathrm{E}[X_k]}_{0} = 0$$

– Therefore:

$$\mathbf{AX} = \mathbf{Y} \sim \mathrm{N}(\mathbf{0}, \delta\mathbf{\Sigma}) \tag{1.28}$$

- The inverse $\mathbf{A}^{-1}$ transforms correlated random variables into uncorrelated ones.

  – Let $Y_i, i = 1, \ldots n$ be correlated multivariate normally distributed random variables with covariance $\delta\mathbf{\Sigma}$:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \text{ with } \mathbf{Y} \sim \mathrm{N}(\mathbf{0}, \delta\mathbf{\Sigma})$$

  * i.e.

$$\mathrm{cov}[Y_i, Y_j] = \delta\Sigma_{ij} \ , \ \ \mathrm{E}\left[Y_i\right] = 0 \ \ \forall i, j = 1, \ldots n$$

  – New random variables:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y} \Longleftrightarrow X_i = \sum_k (A^{-1})_{ik} Y_k$$

* with covariances:

$$
\begin{aligned}
\mathrm{cov}[X_i, X_j] &= \mathrm{cov}[\sum_k (A^{-1})_{ik} Y_k, \sum_m (A^{-1})_{jm} Y_m] \\
&= \sum_k (A^{-1})_{ik} \sum_m (A^{-1})_{jm} \underbrace{\mathrm{cov}[Y_k, Y_m]}_{\delta\Sigma_{km}} \\
&= \sum_k \sum_m (A^{-1})_{ik} \delta\Sigma_{km} (A^{-1})^T_{mj} \\
&= \big(\mathbf{A}^{-1} \delta\mathbf{\Sigma} (\mathbf{A}^{-1})^T\big)_{ij} = \big(\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^T(\mathbf{A}^{-1})^T\big)_{ij}
\end{aligned}
$$

- Every invertible matrix has the property:

$$
(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{1.29}
$$

* Therefore

$$
\mathrm{cov}[X_i, X_j] = \left(\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{\mathbf{1}} \underbrace{\mathbf{A}^T(\mathbf{A}^T)^{-1}}_{\mathbf{1}} \right)_{ij} = \delta_{ij}
$$

– The expectations of the new random variables remain zero:

$$\mathrm{E}[X_i] = \mathrm{E}[\sum_k (A^{-1})_{ik} Y_k] = \sum_k (A^{-1})_{ik} \underbrace{\mathrm{E}[Y_k]}_{0} = 0$$

– Thus, the X_i are uncorrelated and standard normally distributed:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y} \sim \mathrm{N}(\mathbf{0}, \mathbf{1}) \tag{1.30}$$

- Explicit construction of matrix $\mathbf{A}$ iteratively via *Cholesky decomposition*:

$$A_{ji} = \begin{cases} 0 & \text{for } j < i \\ \sqrt{\delta\Sigma_{ii} - \sum_{k=1}^{i-1} A_{ik}^2} & \text{for } j = i \\ \frac{1}{A_{ii}} \left(\delta\Sigma_{ji} - \sum_{k=1}^{i-1} A_{ik}\, A_{jk} \right) & \text{for } j > i \end{cases} \tag{1.31}$$

– with $\delta\Sigma_{ji}$ as given in Equation 1.22:

$$\delta\Sigma_{ji} = \begin{cases} \sigma_i^2 \delta t & \text{for } j = i \\ \rho_{ij}\, \sigma_i \sigma_j \delta t & \text{for } j \neq i \end{cases}$$

- Begin with $i = 1$, $j = 1$ and proceed by solving for A_{j1} for all j.
- Subsequently set $i = 2$ and solve for A_{j2} for all j. Repeat the procedure.
- Illustrated for $n = 3$:

$$A_{11} = \sqrt{\delta\Sigma_{11}}\ , \quad A_{21} = \frac{\delta\Sigma_{21}}{A_{11}} = \frac{\delta\Sigma_{21}}{\sqrt{\delta\Sigma_{11}}}, \quad A_{31} = \frac{\delta\Sigma_{31}}{A_{11}} = \frac{\delta\Sigma_{31}}{\sqrt{\delta\Sigma_{11}}}$$

$$A_{22} = \sqrt{\delta\Sigma_{22} - A_{21}^2} = \sqrt{\delta\Sigma_{22} - \delta\Sigma_{21}^2/\delta\Sigma_{11}}$$

$$A_{32} = \frac{\delta\Sigma_{32} - A_{21}A_{31}}{A_{22}} = \frac{\delta\Sigma_{32} - \delta\Sigma_{21}\delta\Sigma_{31}/\delta\Sigma_{11}}{\sqrt{\delta\Sigma_{22} - \delta\Sigma_{21}^2/\delta\Sigma_{11}}}$$

$$A_{33} = \sqrt{\delta\Sigma_{33} - A_{31}^2 - A_{32}^2} = \sqrt{\delta\Sigma_{33} - \frac{\delta\Sigma_{31}^2}{\delta\Sigma_{11}} - \frac{(\delta\Sigma_{32} - \delta\Sigma_{21}\delta\Sigma_{31}/\delta\Sigma_{11})^2}{\delta\Sigma_{22} - \delta\Sigma_{21}^2/\delta\Sigma_{11}}}$$

* · Finally substitute Equation 1.22 for $\delta\Sigma_{ij}$:

$$
\begin{aligned}
A_{11} &= \sigma_1\sqrt{\delta t} \\
A_{21} &= \sigma_2\sqrt{\delta t}\,\rho_{21}\ , \quad A_{22} = \sigma_2\sqrt{\delta t}\ \sqrt{1-\rho_{21}^2} \\
A_{31} &= \sigma_3\sqrt{\delta t}\,\rho_{31}\ , \quad A_{32} = \sigma_3\sqrt{\delta t}\ \frac{\rho_{32}-\rho_{31}\rho_{21}}{\sqrt{1-\rho_{12}^2}} \\
A_{33} &= \sigma_3\sqrt{\delta t}\ \sqrt{1-\rho_{31}^2-\frac{(\rho_{32}-\rho_{31}\rho_{21})^2}{1-\rho_{12}^2}}
\end{aligned}
$$

* First equation: only a single risk factor
* First *two* equations: for *two* correlated random walks.
* The third and fourth rows: for the *third* correlated random walk.

2

The Variance-Covariance Method

- Vector of risk factors:

$$\mathbf{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{pmatrix}$$

- Linear approximations of the risk factor evolutions throughout (see 1.16):

$$\delta S_i(t) \approx S_i(t)\left[\mu_i \delta t + \delta Z_i\right] \approx S_i(t)\delta Z_i \tag{2.1}$$

- Main idea: Expand portfolio value V in its Taylor series.
  - Portfolio value change $\delta V(\mathbf{S})$ up to second order

$$\begin{aligned}
\delta V(\mathbf{S}(t)) &= V(\mathbf{S}(t) + \delta\mathbf{S}(t)) - V(\mathbf{S}(t)) \\
&\approx \sum_i^n \frac{\partial V}{\partial S_i}\delta S_i(t) + \frac{1}{2}\sum_{i,j}^n \delta S_i(t)\frac{\partial^2 V}{\partial S_i \partial S_j}\delta S_j(t) \\
&= \sum_i^n \Delta_i \delta S_i(t) + \frac{1}{2}\sum_{i,j}^n \delta S_i(t)\Gamma_{ij}\delta S_j(t) \\
&\approx \sum_i^n \widetilde{\Delta}_i \left[\mu_i \delta t + \delta Z_i\right] + \frac{1}{2}\sum_{i,j}^n \left[\mu_i \delta t + \delta Z_i\right]\widetilde{\Gamma}_{ij}\left[\mu_j \delta t + \delta Z_j\right] \\
&\approx \sum_i^n \widetilde{\Delta}_i \,\delta Z_i + \frac{1}{2}\sum_{i,j}^n \delta Z_i \widetilde{\Gamma}_{ij}\,\delta Z_j
\end{aligned} \tag{2.2}$$

- First "≈": broken off the Taylor series for *portfolio value* V after 2nd order terms,
- Second "≈": linear approximation of the *risk factors* S_i
- Third "≈":drift neglected.

- Second Order proxy for δV is called *Delta-Gamma approximation*
- First order for δV is called *Delta approximation*
- Abbreviations Δ_i and Γ_{ij} denote the risk factor *sensitivities* of V:

$$\Delta_i := \frac{\partial V}{\partial S_i} \ , \ \ \Gamma_{ij} := \frac{\partial^2 V}{\partial S_i \partial S_j} \ , \ \ i, j = 1, \dots n$$

- Γ_{ij} (sometimes called *Hessian matrix*) contains also *mixed* partial derivatives.
- The sensitivities usually appear in connection with the *current levels* $S_i(t)$ and $S_j(t)$:

$$\widetilde{\Delta}_i := S_i(t) \frac{\partial V}{\partial S_i} \ , \ \ \widetilde{\Gamma}_{ij} := S_i(t) S_j(t) \frac{\partial^2 V}{\partial S_i \partial S_j} \tag{2.3}$$

- Delta-Gamma Proxy for δV in vector form:

$$\delta V(\mathbf{S}(t)) = \begin{pmatrix} \widetilde{\Delta}_1 & \cdots & \widetilde{\Delta}_n \end{pmatrix} \begin{pmatrix} \delta Z_1 \\ \vdots \\ \delta Z_n \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} \delta Z_1 & \cdots & \delta Z_n \end{pmatrix} \begin{pmatrix} \widetilde{\Gamma}_{1,1} & \cdots & \widetilde{\Gamma}_{1,n} \\ \vdots & \ddots & \vdots \\ \widetilde{\Gamma}_{n,1} & \cdots & \widetilde{\Gamma}_{n,n} \end{pmatrix} \begin{pmatrix} \delta Z_1 \\ \vdots \\ \delta Z_n \end{pmatrix}$$

$$= \widetilde{\mathbf{\Delta}}^T \delta\mathbf{Z} + \frac{1}{2} \delta\mathbf{Z}^T \widetilde{\mathbf{\Gamma}} \delta\mathbf{Z}$$

- Taylor-approximations for a *straddle* (a portfolio made up of a call and a put) are presented in Figure 2.1.

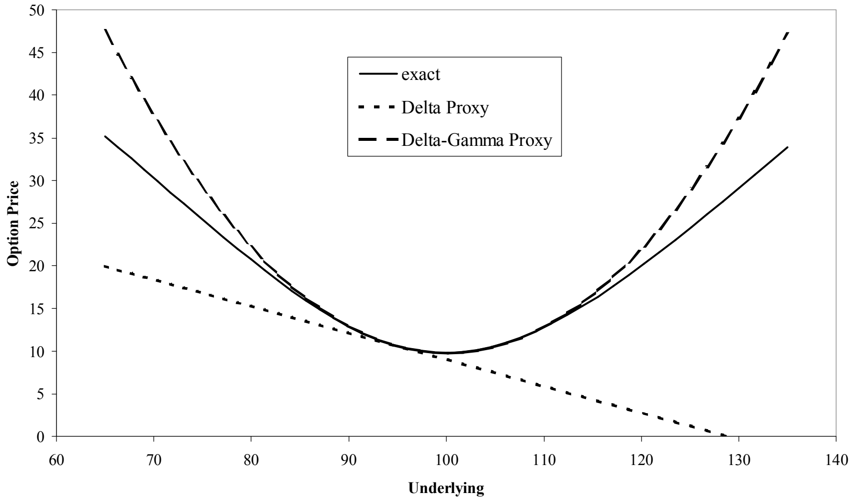

Figure 2.1: Black-Scholes price of a straddle (strike = 100, time to maturity = 1 year) on an underlying S (vol 25%, dividend yield 6%, repo rate 3%). The dashed line is the Delta-Gamma proxy, the dotted line is the simple Delta proxy. The Taylor expansion was done around $S = 95$.

2.1 Portfolios vs. Financial Instruments

- Consider Portfolio with value V consisting of M different financial instruments with values V_k , $k = 1, \dots M$.

$$V(t) = \sum_{k=1}^{M} N_k V_k(\mathbf{S}(t))$$

  where N_k denotes the number of instrument V_k held in the portfolio.

- Change in value $\delta V(t)$ (up to second order)

$$\begin{aligned}
\delta V(\mathbf{S}(t)) &= \sum_{k=1}^{M} N_k \delta V_k(\mathbf{S}(t)) \\
&\approx \sum_{k=1}^{M} N_k \left[\sum_{i}^{n} \frac{\partial V_k}{\partial S_i} \delta S_i(t) + \frac{1}{2} \sum_{i,j}^{n} \delta S_i(t) \frac{\partial^2 V_k}{\partial S_i \partial S_j} \delta S_j(t) \right] \\
&= \sum_{k=1}^{M} N_k \sum_{i}^{n} \Delta_i^k \delta S_i(t) + \frac{1}{2} \sum_{k=1}^{M} N_k \sum_{i,j}^{n} \delta S_i(t) \Gamma_{ij}^k \delta S_j(t)
\end{aligned}$$

– with the sensitivities of the *financial instruments*:

$$\Delta_i^k := \frac{\partial V_k}{\partial S_i} \ , \ \Gamma_{ij}^k := \frac{\partial^2 V_k}{\partial S_i \partial S_j} \ , \ i,j = 1, \ldots n \ ; \ k = 1, \ldots M$$

- Simple rearranging yields the portfolio sensitivities:

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \sum_i^n \delta S_i(t) \sum_{k=1}^M N_k \Delta_i^k + \frac{1}{2} \sum_{i,j}^n \delta S_i(t) \delta S_j(t) \sum_{k=1}^M N_k \Gamma_{ij}^k \\ &= \sum_i^n \delta S_i(t) \Delta_i + \frac{1}{2} \sum_{i,j}^n \delta S_i(t) \delta S_j(t) \Gamma_{ij} \end{aligned}$$

– The sensitivities of the entire *portfolio* are just the sum of the sensitivities of the individual *financial instruments* (*position mapping*):

$$\Delta_i = \sum_{k=1}^M N_k \Delta_i^k \ , \ \Gamma_{ij} = \sum_{k=1}^M N_k \Gamma_{ij}^k$$

- With proxy 2.1 for the risk factors:

$$\begin{aligned}\delta V(\mathbf{S}(t)) &\approx \sum_{i}^{n} \widetilde{\Delta}_i \; [\mu_i \delta t + \delta Z_i] + \frac{1}{2} \sum_{i,j}^{n} \; [\mu_i \delta t + \delta Z_i] \, \widetilde{\Gamma}_{ij} \; [\mu_j \delta t + \delta Z_j] \\ &\approx \sum_{i}^{n} \widetilde{\Delta}_i \, \delta Z_i + \frac{1}{2} \sum_{i,j}^{n} \; \delta Z_i \widetilde{\Gamma}_{ij} \, \delta Z_j\end{aligned}$$

  - with the modified portfolio sensitivities (*VaR mapping)*:

$$\begin{aligned}\widetilde{\Delta}_i &:= S_i(t) \Delta_i = S_i(t) \sum_{k=1}^{M} N_k \Delta_i^k \\ \widetilde{\Gamma}_{ij} &:= S_i(t) S_j(t) \Gamma_{ij} = S_i(t) S_j(t) \sum_{k=1}^{M} N_k \Gamma_{ij}^k\end{aligned}$$

2.2 The Delta-Normal Method

- Taylor series 2.2 of the portfolio value only up to the *linear* term:

$$\begin{aligned}\delta V(\mathbf{S}(t)) &\approx \sum_i^n \frac{\partial V}{\partial S_i}\delta S_i(t) \\ &\approx \sum_i^n \widetilde{\Delta}_i\left[\mu_i \delta t + \delta Z_i\right] \approx \sum_i^n \widetilde{\Delta}_i\,\delta Z_i = \widetilde{\boldsymbol{\Delta}}^T \delta \mathbf{Z}\end{aligned} \tag{2.4}$$

2.2.1 Value at Risk with respect to a Single Risk Factor

- For a single risk factor with the portfolio sensitivity $\Delta := \partial V / \partial S$:

$$\delta V(S(t)) \approx \Delta \delta S(t) \tag{2.5}$$

- A portfolio with a linear sensitivity Δ can be intuitively interpreted as a portfolio consisting of Δ risk factors.
  - The value at risk is thus given by Equation 1.15.

- However, one does not know, a priori, whether Δ is greater than or less than 0.
- Equation 1.15 holds with the correspondence $\Delta \widehat{=} N$ for $\Delta > 0$ and $\Delta \widehat{=} -N$ for $\Delta < 0$

- In any case: V is linear and in consequence, a monotonous function of S. Therefore, in the sense of Equation 1.5

$$\begin{aligned} \mathrm{VaR}_V(c) \approx \max\{ & \widetilde{\Delta}\left[1 - \exp\left(\mu\delta t + Q_{1-c}\sigma\sqrt{\delta t}\right)\right], \\ & \widetilde{\Delta}\left[1 - \exp\left(\mu\delta t - Q_{1-c}\sigma\sqrt{\delta t}\right)\right]\} \end{aligned} \tag{2.6}$$

  - As usual: $\widetilde{\Delta} := S(t)\Delta$ and Q_{1-c} is the $(1-c)$-percentile of the standard normal distribution.

- The maximum function in Equation 1.5 effects the correct choice for the VaR.
  - For $\Delta > 0$, the *lower bound* of the confidence interval of the risk factor is relevant

  - For $\Delta < 0$, the *upper bound* of the confidence interval of the risk factor is relevant.

- Now: Approximate the *risk factor* change with Equation 2.1:

$$\begin{aligned}
\mathrm{VaR}_V(c) &\approx \max\{\widetilde{\Delta}\left[-\mu\delta t - Q_{1-c}\sigma\sqrt{\delta t}\right], \widetilde{\Delta}\left[-\mu\delta t + Q_{1-c}\sigma\sqrt{\delta t}\right]\} \\
&= \max\left\{-\widetilde{\Delta}Q_{1-c}\sigma\sqrt{\delta t}, +\widetilde{\Delta}Q_{1-c}\sigma\sqrt{\delta t}\right\} - \widetilde{\Delta}\mu\delta t \\
&= \left|\widetilde{\Delta}Q_{1-c}\sigma\sqrt{\delta t}\right| - \widetilde{\Delta}\mu\delta t \qquad (2.7)
\end{aligned}$$

  - In this proxy, the maximum function produces precisely the absolute value of the risk which is caused by the volatility of the risk factor.
  - A positive drift μ of the risk factor
    * reduces the portfolio risk when $\widetilde{\Delta} > 0$
    * *increases* the portfolio risk when $\widetilde{\Delta} < 0$

- We were able to deduce information about the *unknown* distribution of the *portfolio's value* V from the *known* distribution of the *risk factor* S!

2.2.2 The Value at Risk with respect to Several Risk Factors

- Avoidance of the *unknown* distribution of V is only possible within the roughest approximation in 2.1:

$$\delta S_i(t) \approx S_i(t)\delta Z_i$$

- Squaring both sides of 2.7 with $\mu = 0$ yields:

$$\begin{aligned}
\mathrm{VaR}_V^2(c) &\approx \widetilde{\Delta}^2 \left(Q_{1-c}\right)^2 \sigma^2 \delta t \\
&= \Delta^2 \left(Q_{1-c}\right)^2 S(t)^2 \sigma^2 \delta t \\
&= \Delta^2 \left(Q_{1-c}\right)^2 S(t)^2 \mathrm{var}\left[\delta Z\right] \\
&= \Delta^2 \left(Q_{1-c}\right)^2 \mathrm{var}\left[\delta S(t)\right]
\end{aligned}$$

- On the other hand, the variance of V is

$$\mathrm{var}\left[\delta V\right] \approx \mathrm{var}\left[\Delta \delta S(t)\right] = \Delta^2 \mathrm{var}\left[\delta S(t)\right]$$

- In this approximation the (square of the) *Value at Risk* of the portfolio can be expressed in terms of the *variance* of the portfolio:

$$\mathrm{VaR}_V^2(c) \approx (Q_{1-c})^2 \,\mathrm{var}\left[\delta V\right]$$

- Now, only the *variance* of the portfolio's value needs to be determined *not* its entire *distribution* nor its *percentiles*!

- This is also true for several risk factors in the approximation 1.24, i.e.

$$\delta V \approx \sum_{i=1}^{n} \widetilde{\Delta}_i \delta Z_i \Longrightarrow \mathrm{VaR}_V(c) \approx Q_{1-c}\sqrt{\mathrm{var}\left[\delta V\right]}$$

- The variance of a sum of random variables is simply the sum of the covariances of these random variables:

$$\begin{aligned}
\text{var}\left[\delta V\right] &\approx \sum_{i,j=1}^{n} \widetilde{\Delta}_i \widetilde{\Delta}_j \text{cov}\left[\delta Z_i, \delta Z_j\right] \\
&= \sum_{i,j=1}^{n} \widetilde{\Delta}_i \delta \Sigma_{ij} \widetilde{\Delta}_j \qquad (2.8) \\
&= \delta t \sum_{i,j=1}^{n} \widetilde{\Delta}_i \sigma_i \rho_{ij} \sigma_j \widetilde{\Delta_j}
\end{aligned}$$

- Thus the Value at Risk is

$$\begin{aligned}
\text{VaR}_V(c) &\approx Q_{1-c} \sqrt{\text{var}\left[\delta V\right]} \\
&= Q_{1-c} \sqrt{\widetilde{\boldsymbol{\Delta}} \delta \boldsymbol{\Sigma} \widetilde{\boldsymbol{\Delta}}} \\
&= Q_{1-c} \sqrt{\delta t} \sqrt{\sum_{i,j=1}^{n} \widetilde{\Delta}_i \sigma_i \rho_{ij} \sigma_j \widetilde{\Delta}_j} \qquad (2.9)
\end{aligned}$$

- This is the central equation for the Delta-Normal method.
  - If all portfolio sensitivities are non-negative,the portfolio-VaR can be expressed in terms of the VaRs w.r.t. the individual risk factors:

$$\mathrm{VaR}_V(c) \approx \sqrt{\sum_{i,j=1}^{n} \mathrm{VaR}_i(c)\rho_{ij}\mathrm{VaR}_j(c)} \tag{2.10}$$

- The mean return can be introduced into approximation 2.9 after the fact analogously to the single risk factor case:

$$\mathrm{VaR}_V(c) \approx Q_{1-c}\sqrt{\delta t}\sqrt{\sum_{i,j=1}^{n} \widetilde{\Delta}_i\sigma_i\rho_{ij}\sigma_j\widetilde{\Delta}_j} - \delta t\sum_i \widetilde{\Delta}_i\,\mu_i \tag{2.11}$$

  - The VaR of a portfolio containing holdings N_i of the risk factors themselves is a special case:

$$\mathrm{VaR}_V(c) \approx Q_{1-c}\sqrt{\delta t}\sqrt{\sum_{i,j=1}^{n} N_iS_i\sigma_i\rho_{ij}\sigma_jN_jS_j} - \delta t\sum_i N_iS_i\,\mu_i \tag{2.12}$$

- Summary of the Delta-Normal approach to the calculation of the Value at Risk
  - Calculate of the sensitivities Δ_i of the portfolio with respect to all risk factors S_i.
  - Multiply the covariance matrix $\delta\mathbf{\Sigma}$ with the portfolio sensitivities Δ_i and the current risk factor values S_i to obtain the portfolio variance as in 2.8.
  - The covariance matrix's elements are products of the risk factor volatilities and correlations as in 1.22.
  - Multiply the portfolio variance as in Equation 2.9 with the liquidation period and the square of the percentile corresponding to the desired confidence interval (for example, 2.326 for 99% confidence).
  - The square root of the thus obtained number is the Value at Risk of the entire portfolio, neglecting the effect of the risk factor drifts.
  - The effect of the mean yields can be taken into consideration as in Equation 2.11.

2.3 Remarks on Volatility and Correlation

- Price change δV of a financial instrument or portfolio with a sensitivity Δ_i with respect to a single risk factor S_i in delta-normal approximation

$$\delta V(t) \approx \Delta_i(t)\, \delta S_i(t) \approx \Delta_i(t)\, S_i(t)\, \delta \ln\left(S_i(t)\right) \tag{2.13}$$

  - The second step follows from the linear approximation 1.16 of the risk factor change[1]:

$$\delta S(t) \approx S(t)\, \delta \ln S(t) \quad \text{for} \quad \delta S(t) \ll S(t) \tag{2.14}$$

[1] This can also be shown from first principles:

$$\begin{aligned}
\delta \ln S(t) \equiv \ln S(t+\delta t) - \ln S(t) &= \ln\left(\frac{S(t+\delta t)}{S(t)}\right) \\
&= \ln \frac{S(t) + [\overbrace{S(t+\delta t) - S(t)}^{\delta S(t)}]}{S(t)} \\
&= \ln\left(1 + \frac{\delta S(t)}{S(t)}\right) = \frac{\delta S(t)}{S(t)} - \frac{1}{2}\left(\frac{\delta S(t)}{S(t)}\right)^2 \pm \cdots \approx \frac{\delta S(t)}{S(t)}
\end{aligned}$$

  - The linear approximation of $\delta \ln V$ is likewise equal to the *relative* change $\delta V/V$.

- Dividing 2.13 by V yields the relationship between the *logarithmic* changes of V and S:

$$\delta \ln \left(V(t)\right) \approx \frac{\delta V(t)}{V(t)} \approx \frac{\Delta_i(t)}{V(t)}\, S_i(t)\, \delta \ln \left(S_i(t)\right) \tag{2.15}$$

  - At time t, all variables appearing are known except for the changes δV and $\delta \ln(S)$. The variance of this equation is thus

$$\operatorname{var}\left[\delta \ln \left(V(t)\right)\right] \approx \frac{\Delta(t)^2}{V(t)^2}\, S_i(t)^2 \operatorname{var}\left[\delta \ln \left(S_i(t)\right)\right]$$

  - Expressing the variances through the volatilities, we obtain the relationship between the volatility σ_V of the price of a financial instrument or portfolio and the volatility σ_i of a risk factor:

$$\sigma_V(t) = \left| \frac{\Delta_i(t)}{V(t)}\, S_i(t) \right| \sigma_i(t) \tag{2.16}$$

- The absolute value indicates explicitly that the volatility, being the square root of the variance, is always positive.

- The correlation of *another* risk factor S_j with a portfolio dependent on S_i is the same as its correlation with the risk factor S_i itself:

$$\begin{aligned}
&\sigma_V\, \sigma_j\, \rho_{V\,j}\, \delta t \\
&= \mathrm{cov}\left[\delta \ln\left(V\right),\ \delta \ln\left(S_j\right)\right] = \mathrm{cov}\left[\frac{\Delta_i}{V} S_i\, \delta \ln\left(S_i\right),\ \delta \ln\left(S_j\right)\right] \\
&= \frac{\Delta_i}{V} S_i\, \mathrm{cov}\left[\delta \ln\left(S_i\right),\ \delta \ln\left(S_j\right)\right] = \underbrace{\frac{\Delta_i}{V} S_i\, \sigma_i}_{\sigma_V}\, \sigma_j\, \rho_{ij}\, \delta t \\
&= \sigma_V\, \sigma_j\, \rho_{ij}\, \delta t
\end{aligned}$$

  - and hence

$$\rho_{V\,j} = \rho_{ij} \tag{2.17}$$

3

Interest Rate Risk

- Pricing is done by discounting cash flows or more generally by discounting future expectations.
  - Thus the yield of any zero bond can influence a price and is therefore a risk factor.
  - This is *interest rate risk*.

- To quantify this, one needs
  * either the volatilities of the discount factors (volatilities of zero bond *prices*)
  * or the volatilities of the spot rates (volatilities of zero bond *yields*)

- The relation between *yield volatilities* and *price volatilities* is a special case of Equation 2.16.
  - The relative sensitivity Δ/V for bonds is, as is well known, the (negative) *modified duration*:

$$
\begin{aligned}
\sigma_B(t,T) &= D_{\text{mod}} R(t,T)\, \sigma_R(t,T) \\
&= \begin{cases} \frac{R(t,T)(T-t)}{1+R(t,T)} \sigma_R(t,T) & \text{Discrete Compounding} \\ \frac{R(t,T)(T-t)}{1+R(t,T)(T-t)} \sigma_R(t,T) & \text{Simple Compounding} \end{cases}
\end{aligned}
\tag{3.1}
$$

3.1 Interpolations

- In principle: at time t infinetely many spot rates $R(t,T)$

  - This infinite number of risk factors is estimated from a *finite* number (15, say) of spot rates observed in the market.
  - These observed spot rates are called *vertices.*

- Let T_i, $i = 1, ..., n$ denote the maturities corresponding to these vertices.
- A spot rate $R(t, T)$ at time t for a maturity T between two vertices T_i and T_{i-1} is estimated by an *interpolation.*

  - There are many plausible ways of constructing such an interpolation, for example, linear interpolation, splines, etc.
  - The simplest: straight line between neigboring points defined by the interest rates at the vertices (*linear interpolation*). The spot rate with maturity T is then

$$R(t, T) = \frac{T_i - T}{T_i - T_{i-1}} R(t, T_{i-1}) + \frac{T - T_{i-1}}{T_i - T_{i-1}} R(t, T_i)$$
$$\text{with } t = T_0 < T_1 < \cdots < T_{i-1} \leq T < T_i < \cdots < T_n \qquad (3.2)$$

- From this interpolated $R(t,T)$ the discount factor $B_R(t,T)$ is calculated using any compounding convention.
  - Thus, the discount factor itself need not be estimated by means of an interpolation.
- For VaR calculations, we need the price volatility of the discount factor.
  - The price volatility is by definition, the standard deviation of the logarithmic price changes.
  - The logarithmic price changes are just the yields the discount factors, i.e. they correspond to the spot rates.
  - To be consistent, the *same* interpolation must used for the price volatilities of the discount factors as for the spot rates.
  - If linear interpolation 3.2 for the spot rates, then the price volatilities of the zero bonds must also be linearly interpolated:

$$\sigma(t,T) = \frac{T_i - T}{T_i - T_{i-1}}\sigma(t,T_{i-1}) + \frac{T - T_{i-1}}{T_i - T_{i-1}}\sigma(t,T_i) \tag{3.3}$$

3.2 Cash Flow Mapping

- Interpolations can be used to construct market parameters (risk factors) in any situation.
- This is sufficient for the pricing but *not* for risk management.
- In risk management, not only the spot rate itself but also its volatility *and* its correlations with all other spot rates are needed.
  - For risk management, a maturity T_i qualifies as a vertex only if
    - the spot rate (or dicount factor)
    - *and* its volatility
    - *and* its correlation with all other vertices are given.
  - Financial data providers generally make spot rates and their volatilities and correlations available daily at the following vertices:

1m	3m	6m	12m	2y	3y	4y	5y	7y	9y	10y	15y	20y	30y

- However: correlations cannot straighfowardly be “interpolated”.

- To overcome this problem, we take the opposite approach:
  - Instead of interpolating to fit market parameters to a given instrument (cash flows),
  - we fit the instrument to the given market parameters, i.e. to the given vertices, by means of *cash flow mapping*.
- A cash flow $C(T)$ occurring at time T between two vertices T_{i-1} and T_i is divided into two artificial cash flows occurring at the previous vertex T_{i-1} and at the following vertex T_i.
  - These "artificial cash flows" are expressed in terms of their (percentual) share a and b of the original cash flow:

$$C(T) \rightarrow \begin{cases} C(T_{i-1}) = a\,C(T) \\ C(T_i) = b\,C(T) \end{cases}$$
$$\text{where} \quad t < T_1 < \cdots < T_{i-1} \leq T < T_i < \cdots < T_n \tag{3.4}$$

  - The proportions a and b are determined uniquely through two conditions.

- One condition is obvious: *The present value of the cash flow should remain unchanged under the mapping.*

  - This invariance of the present value yields the following condition

$$\begin{aligned} B_R(t,T)C(T) &= B_R(t,T_{i-1})C(T_{i-1}) + B_R(t,T_i)C(T_i) \\ &= B_R(t,T_{i-1})\, a\, C(T) + B_R(t,T_i)\, b\, C(T) \end{aligned}$$

    Thus:

$$B_R(t,T) = aB_R(t,T_{i-1}) + bB_R(t,T_i) \tag{3.5}$$

- There are several plausible choices for the second conditoin.

3.2.1 Risk-Based Cash Flow Mapping

- The second condition is chosen as: *The risk should remain unchanged under the mapping.*

- The right hand side in 3.5 can be thought of as the value of a portfolio composed of two risk factors which are represented by the *discount factors* with maturities T_i and T_{i-1}.
- Using the discount factors (and not the spot rates) directly as the risk factors means: the portfolio is *linear* in the risk factors.
  - Then the *risk* of a portfolio is propotional to the portfolio's *variance* (see Section 2.2.2).
  - This is only possible when the stochastic evolution of the risk factors is approximated as in 1.24.
  - With this approximation, the variance of $B_R(t,T_i)$ is simply $B_R(t,T_i)^2\sigma_i^2\delta t$, where σ is the *price volatility* of $B_R(t,T_i)$.
- Analogous to Equation 2.8, the variance of this portfolio is then

$$\begin{aligned} B_R(t,T)^2\sigma_T^2\delta t &= a^2 B_R(t,T_{i-1})^2\sigma_{i-1}^2\delta t \\ &+ b^2 B_R(t,T_i)^2\sigma_i^2\,\delta t \\ &+ 2abB_R(t,T_{i-1})B_R(t,T_i)\,\rho_{i,i-1}\,\sigma_{i-1}\,\sigma_i\delta t \end{aligned} \tag{3.6}$$

- The price volatilities σ_i, σ_{i-1} and the price correlation $\rho_{i,i-1}$ are known.
- σ_T is obtained by an interpolation consistent with the interpolation used for the spot rates.
  - for example, by the linear interpolation 3.3 if Equation 3.2 was used for the spot rates.
- Thus, only the two weights a and b in Equation 3.6 are unknown.

• Equations 3.5 and 3.6 completely determine a and b.

- The abbreviations

$$\alpha = \frac{B_R(t, T_{i-1})}{B_R(t, T)} a \, , \quad \beta = \frac{B_R(t, T_i)}{B_R(t, T)} b \tag{3.7}$$

simplifies the two conditions substantially:

$$\begin{aligned} 1 &= \alpha + \beta \\ \sigma_T^2 &= \alpha^2 \sigma_{i-1}^2 + \beta^2 \sigma_i^2 + 2\alpha\beta \, \rho_{i,i-1} \, \sigma_i \, \sigma_{i-1} \end{aligned}$$

- Substituting $\beta = 1 - \alpha$ into the second condition yields a *quadratic* equation with only one unknown:

$$\sigma_T^2 = \alpha^2(\sigma_{i-1}^2 + \sigma_i^2 - 2\rho_{i,i-1}\,\sigma_i\,\sigma_{i-1}) + 2\alpha\,(\rho_{i,i-1}\,\sigma_i\,\sigma_{i-1} - \sigma_i^2) + \sigma_i^2$$

  - with the solution

$$\alpha = \frac{\sigma_i^2 - \rho_{i,i-1}\,\sigma_i\,\sigma_{i-1} \pm \sqrt{\sigma_T^2(\sigma_i^2 + \sigma_{i-1}^2 - 2\rho_{i,i-1}\,\sigma_i\,\sigma_{i-1}) - \sigma_i^2\sigma_{i-1}^2(1 - \rho_{i,i-1}^2)}}{\sigma_i^2 + \sigma_{i-1}^2 - 2\rho_{i,i-1}\,\sigma_i\,\sigma_{i-1}} \tag{3.8}$$

- This α (and also $\beta = 1 - \alpha$) yields via Equation 3.7 the desired proportionality factors a and b.

  - The mapping in Equation 3.4 is then explicitly

$$C(T) \rightarrow \begin{cases} C(T_{i-1}) = \alpha\frac{B_R(t,T)}{B_R(t,T_{i-1})}C(T) \\ C(T_i) = (1-\alpha)\frac{B_R(t,T)}{B_R(t,T_i)}C(T) \end{cases}$$
$$\text{with } \; t = T_0 < T_1 < \cdots < T_{i-1} \leq T < T_i < \cdots < T_n \tag{3.9}$$

- Note that the weights a and b do *not* add up to one (but α and β do).

- At no point did the derivation require knowledge of the interpolation method used to estimate the risk factor or its volatility at time T .
  - The cash flow mapping holds for any arbitrary interpolation.
  - Only in concrete application is it necessary to decide on an interpolation method, since one needs the interpolated values $B_R(t, T)$ and σ_T in Equations 3.8 und 3.9

3.2.2 Duration-Based Cash Flow Mapping

- Now, the second condition is chosen more traditionally as: *The (Macaulay) duration should remain unchanged under the mapping.*
- The Macaulay duration of a zero bond is simply its lifetime (for all forms of compounding).

- The condition is therefore given by

$$T - t = \frac{(T_{i-1} - t)B_R(t, T_{i-1})a + (T_i - t)B_R(t, T_i)b}{B_R(t, T)} \tag{3.10}$$

- This, together with the present value condition 3.5, gives the two conditions:

$$1 = \alpha + \beta$$
$$T - t = (T_{i-1} - t)\alpha + (T_i - t)\beta$$

  with α and β as defined in 3.7.

- Substituting $\beta = 1 - \alpha$ into the second condition yields a *linear* equation for α which is easily solved:

$$\alpha = \frac{T_i - T}{T_i - T_{i-1}} \Rightarrow \beta = 1 - \alpha = \frac{T - T_{i-1}}{T_i - T_{i-1}}$$

- This looks lie a linear interpolation as in Equation 3.2.
  - However, the cash flow can not be distributed linearly on the vertices since the relevant weightsare actually a and b and *not* α and β.

- The correct duration-based cash flow mapping is

$$
\begin{aligned}
C(T) \rightarrow &\left\{ \begin{array}{r} C(T_{i-1}) = \frac{T_i - T}{T_i - T_{i-1}} \frac{B_R(t,T)}{B_R(t,T_{i-1})} C(T) \\ C(T_i) = \frac{T - T_{i-1}}{T_i - T_{i-1}} \frac{B_R(t,T)}{B_R(t,T_i)} C(T) \end{array} \right. \\
\text{where} \quad & t = T_0 < T_1 < \cdots < T_{i-1} \leq T < T_i < \cdots < T_n
\end{aligned} \tag{3.11}
$$

4

Currency Rebasing of Volatilities and Correlations

4.1 Rebasing the Volatility

- Prices (as opposed to interest rates)are always quoted with reference to a currency.

- Volatilities and correlations of prices depend heavily on the currency of the price.
  - A price in a currency which is not the *reference currency* or *base currency* entails two risks:
    - the risk of value changes in the quoted currency
    - risk of exchange rate changes with respect to the reference currency.
  - This is similar (but not the same!) as having a portfolio with two risk factors.
    - first risk factor: the price in the quoted currency
    - second risk factor: the exchange rate with the reference currency.
- Let S be the price of a risk factor in a "foreign" currency
- Let D be the exchange rate with respect to the reference currency
  - D units of the reference currency are received for one unit of the foreign currency.
- The price at time t of the risk factor in the reference currency is simply $D(t)S(t)$.

- The variance of the logarithmic change of this value is:

$$
\begin{aligned}
&\text{var}\left[\delta \ln\left(D(t)S(t)\right)\right] \\
&= \text{var}\left[\underbrace{\delta \ln(D(t))}_{\text{Random Variable } X_1} + \underbrace{\delta \ln\left(S(t)\right)}_{\text{Random Variable } X_2}\right] = \sum_{a,b=1}^{2} \text{cov}\left[X_a, X_b\right] \\
&= \text{var}\left[\delta \ln\left(D(t)\right)\right] + \text{var}\left[\delta \ln\left(S(t)\right)\right] + 2\text{cov}\left[\delta \ln\left(D(t)\right), \delta \ln\left(S(t)\right)\right]
\end{aligned}
$$

  - Expressing these variances in terms of volatilities and correlations yields

$$
\sigma_{DS} = \sqrt{\sigma_D^2 + \sigma_S^2 + 2\rho_{S,D}\,\sigma_D\,\sigma_S} \tag{4.1}
$$

  - This is the volatility transformation.
    * on the left-hand side: the desired volatility of S in the reference currency
    * on the right-hand side: the volatility of S in the foreign currency, the volatility of the exchange rate and the correlation between them.

- Compare to Equation 2.12.

- The Square root in 2.12 is the volatility of a *sum* of risk factors.
- Equation 4.1 is the volatility of a *product* of risk factors (the risk factors are D and S)
- The expressions are similar, but there are two differences:
  * In contrast to Equation 2.12, no approximation like 2.1 of the risk factor evolution is necessary for Equation 4.1. It is therefore an exact equality.
  * In Equation 4.1 the current levels of the risk factors D and S do *not* appear.

4.1.1 Example

- British bond with a face value of $N =$GBP $100,000$ whose price and volatility is quoted in British Pounds (GBP).
- The value at risk *in GBP* is simply the product of
  - the daily **volatility** $\sigma = 0.026\%$,

- the **current level** 99.704% of the face value N,
- the square root of the **liquidation period**,
- and the **percentile** for the confidence level c:

$$\begin{aligned}\mathrm{VaR}(c,t,T) &= Q_{1-c}\sqrt{T-t}\; S\, N\, \sigma_S \\ &= Q_{1-c}\sqrt{T-t} \times 97.704\% \ \times N \times 0.00026 \\ &= Q_{1-c}\sqrt{T-t} \times 25.38\,\mathrm{GBP}\end{aligned}$$

- What is the value at risk for a USD based investor?
- A further risk factor must be taken into consideration, the USD/GBP exchange rate D.
  - This value and its daily volatility are: $D = 1.469$ USD/GBP and $\sigma_D = 0.404\%$.
  - The correlation between the two risk factors is $\rho_{S,D} = -0.11574$.
- The value at risk in USD is (as above) the product of

- the daily **volatility** in USD: σ_{DS} from 4.1,
- the **current level** in USD: $D(t)S(t)N$,
- the square root of the **liquidation period**,
- and the **percentile** for the confidence level c:

$$\begin{aligned}\text{VaR}(c,t,T) &= Q_{1-c}\sqrt{T-t}D(t)S(t)N\sigma_{SD} \\ &= Q_{1-c}\sqrt{T-t}D(t)S(t)N\sqrt{\sigma_D^2+\sigma_S^2+2\rho_{S,D}\,\sigma_D\,\sigma_S} \\ &= Q_{1-c}\sqrt{T-t}\times 97,704\,\text{GBP}\times 1.469\frac{\text{USD}}{\text{GBP}} \\ &\times\sqrt{0.00404^2+0.00026^2-2\times 0.11574\times 0.00404\times 0.00026} \\ &= Q_{1-c}\sqrt{T-t}\times 577.39\,\text{USD}\end{aligned}$$

- Thus, the risk of the position mainly stems from the foreign currency (as seen from USD).
  - The price risk of the bond with respect to its home currency only plays a minor role.

4.2 Rebasing Correlations

- The transformation for correlations is more complex since *two* risk factors are involved.
- We must distinguish two cases:
  - One risk factor is given in the reference currency and the other in the "foreign" currency.
  - Both risk factors are quoted in a foreign currency.

Case 1

- Risk factor price S_1 is given in the foreign currency.
- Risk factor price S_2 is given in the reference currency.
- Exchange rate: D units of the reference currency are received for one unit in the foreign currency.
- The price of S_1 in the reference currency is DS_1.

- Consider the covariance of the logarithmic price changes in the reference currency:

$$\begin{aligned}&\operatorname{cov}\left[\delta \ln\left(D\, S_1\right),\ \delta \ln\left(S_2\right)\right]\\&= \operatorname{cov}\left[\delta \ln\left(D\right) + \delta \ln\left(S_1\right),\ \delta \ln\left(S_2\right)\right]\\&= \operatorname{cov}\left[\delta \ln\left(D\right),\ \delta \ln\left(S_2\right)\right] + \operatorname{cov}\left[\delta \ln\left(S_1\right),\ \delta \ln\left(S_2\right)\right]\end{aligned}$$

- Express the covariances in terms of volatilities and correlations:

$$\rho_{DS_1,\,S_2}\sigma_{DS_1}\,\sigma_{S_2} = \rho_{D,\,S_2}\,\sigma_D\,\sigma_{S_2} + \rho_{S_1,\,S_2}\,\sigma_{S_1}\,\sigma_{S_2}$$

- Applying 4.1 for the volatility of the product DS_1 finally yields the transformation

$$\rho_{DS_1,\,S_2} = \frac{\rho_{D,\,S_2}\,\sigma_D\,\sigma_{S_2} + \rho_{S_1,\,S_2}\,\sigma_{S_1}\,\sigma_{S_2}}{\sigma_{S_2}\sqrt{\sigma_D^2 + \sigma_{S_1}^2 + 2\,\rho_{D,\,S_1}\,\sigma_D\,\sigma_{S_1}}} \tag{4.2}$$

  - special case with $S_1 := S$ and S_2 set equal to D: correlation of a price S given in the foreign currency and the exchange rate D itself

$$\rho_{DS,\,D} = \frac{\sigma_D\ + \rho_{D,S}\,\sigma_S}{\sqrt{\sigma_D^2 + \sigma_S^2 + 2\,\rho_{D,\,S}\,\sigma_D\,\sigma_S}} = \frac{\sigma_D + \rho_{D,S}\,\sigma_S}{\sigma_{DS}} \tag{4.3}$$

where Equation 4.1 was used.

Case 2

- The risk factor with a price S_1 is quoted in a foreign currency.
- The risk factor S_2 is likewise given in a foreign currency.
- These two foreign currencies may be different.
  - The foreign currency associated with S_1 has exchange rate D_1 with respect to the reference currency.
    * The price of the first risk factor in the reference currency is $D_1 S_1$.
  - The second foreign currency has exchange rate D_2 with respect to the reference currency.
    * The price of the second risk factor in the reference currency is $D_2 S_2$.

- Consider the covariance of the logarithmic changes of the prices

$$\begin{aligned}
&\text{cov}\left[\delta \ln \left(D_1 S_1\right), \delta \ln \left(D_2 S_2\right)\right] \\
&= \text{cov}\left[\delta \ln(D_1) + \delta \ln\left(S_1\right), \delta \ln(D_2) + \delta \ln\left(S_2\right)\right] \\
&= \text{cov}\left[\delta \ln(D_1), \delta \ln(D_2)\right] + \text{cov}[\delta \ln\left(S_1\right), +\delta \ln\left(S_2\right)] \\
&+ \text{cov}\left[\delta \ln(D_1), \delta \ln\left(S_2\right)\right] + \text{cov}[\delta \ln\left(S_1\right), \delta \ln(D_2)]
\end{aligned}$$

- Express the covariances in terms of volatilities and correlations:

$$\begin{aligned}
\rho_{D_1 S_1, D_2 S_2} \sigma_{D_1 S_1} \sigma_{D_2 S_2} = {} & \rho_{D_1, D_2} \sigma_{D_1} \sigma_{D_2} + \rho_{D_1, S_2} \sigma_{D_1} \sigma_{S_2} \\
& + \rho_{D_2, S_1} \sigma_{D_2} \sigma_{S_1} + \rho_{S_1, S_2} \sigma_{S_1} \sigma_{S_2}
\end{aligned}$$

- Equation 4.1 for the volatilities of the products $D_1 S_1$ and $D_2 S_2$ finally yields the transformation:

$$\begin{aligned}
&\rho_{D_1 S_1, D_2 S_2} \\
&= \frac{\rho_{D_1, D_2} \sigma_{D_1} \sigma_{D_2} + \rho_{D_1, S_2} \sigma_{D_1} \sigma_{S_2} + \rho_{D_2, S_1} \sigma_{D_2} \sigma_{S_1} + \rho_{S_1, S_2} \sigma_{S_1} \sigma_{S_2}}{\sqrt{\sigma_{D_1}^2 + \sigma_{S_1}^2 + 2\, \rho_{D_1, S_1} \sigma_{D_1} \sigma_{S_1}} \sqrt{\sigma_{D_2}^2 + \sigma_{S_2}^2 + 2\, \rho_{D_2, S_2} \sigma_{D_2} \sigma_{S_2}}}
\end{aligned} \tag{4.4}$$

- Frequently encountered special case: $D_1 = D_2 = D$.
  - In this case the transformation reduces to

$$\rho_{DS_1,\,DS_2} = \frac{\sigma_D^2 + (\rho_{D,\,S_2}\sigma_{S_2} + \rho_{D,\,S_1}\sigma_{S_1})\,\sigma_D + \rho_{S_1,\,S_2}\sigma_{S_1}\,\sigma_{S_2}}{\sqrt{\sigma_D^2 + \sigma_{S_1}^2 + 2\,\rho_{D,\,S_1}\,\sigma_D\,\sigma_{S_1}}\,\sqrt{\sigma_D^2 + \sigma_{S_2}^2 + 2\,\rho_{D,\,S_2}\,\sigma_D\,\sigma_{S_2}}} \tag{4.5}$$

4.3 Exchange Rates and Cross Rates

- Exchange rates quoted in the data sets supplied by some data providers are usually stated by reference to a base currency, e.g. USD.
- An investor whose reference currency is, for example, EUR must convert the listed exchange rate volatilities and correlations from USD to EUR as the *base currency* before they can be used for risk management.
  - Here not only the risk factors are to be rebased.

  - Also a transformation from one *base* currency (USD) into another *base* currency (EUR) is necessary.

- Here, *cross rates* come into play.
  - For a USD-based investor, the Yen/EUR exchange rate, for example, is a cross rate.
  - Cross rates are obtained by dividing the desired rate (e.g. USD/JPY) by the exchange rate of the new base currency (e.g. USD/EUR).
  - Examples:
    * Exchange rates (w.r.t. USD):

$$D_i(t) = \frac{USD}{EUR}\,, \quad D_j(t) = \frac{USD}{GBP}$$
$$D_n(t) = \frac{USD}{JPY}\,, \quad D_m(t) = \frac{USD}{SFR}$$

    * From these, cross rates for arbitrary combinations can be established,

fro example:

$$D_{ji}(t) = \frac{D_j(t)}{D_i(t)} = \frac{EUR}{GBP}$$
$$D_{ni}(t) = \frac{D_n(t)}{D_i(t)} = \frac{EUR}{JPY}$$
$$D_{mn}(t) = \frac{D_j(t)}{D_n(t)} = \frac{JPY}{SFR}\,, \text{ etc.}$$

- Thus, *quotients* of risk factors (exchange rates) must now be considered.
  - The quotient of two values a and b can, of course, be seen as the product of a and $1/b$.
  - The arguments of the previous section can then be applied.
- Consider the volatilities and correlations of an "inverse risk factor" (exchange

rate) $1/D$:

$$\begin{aligned}\text{var}\left[\delta\,\ln\left(\frac{1}{D(t)}\right)\right] &= \text{var}\left[-\delta\,\ln(D(t))\right] = (-1)^2\text{var}\left[\delta\,\ln(D(t))\right] \\ \text{cov}\left[\delta\,\ln\left(\frac{1}{D(t)}\right), \delta\,\ln(S(t))\right] &= \text{cov}\left[-\delta\,\ln(D(t)), \delta\,\ln(S(t))\right] \\ &= -\text{cov}\left[\delta\,\ln(D(t)), \delta\,\ln(S(t))\right]\end{aligned}$$

- $1/D$ has the same volatility as D
- The correlations of $1/D$ with any risk factor S is just the negative correlation of D with S:

$$\sigma_{\frac{1}{D}} = \sigma_D \quad , \qquad \rho_{\frac{1}{D},S} = -\rho_{D,S} \quad , \qquad \rho_{\frac{1}{D},\frac{1}{S}} = \rho_{D,S} \tag{4.6}$$

- When applying the above equations associate D_j with a risk factor S quoted in the foreign currency and $1/D_b$ with the exchange rate D from that foreign currency to the base currency.

- From Equation 4.1 (with the substitution $S = D_j$ and $D = 1/D_b$), the volatility of a cross rate is:

$$\sigma_{jb} = \sqrt{\sigma_j^2 + \sigma_b^2 - 2\rho_{j,b}\,\sigma_j\,\sigma_b} \tag{4.7}$$

- From Equation 4.4 (with the substitution $S_1 = D_j$, $D_1 = 1/D_i$, $S_2 = D_n$ and $D_2 = 1/D_m$), the correlation between two cross rates $D_{ji} = D_j/D_i$ and $D_{nm} = D_n/D_m$ is:

$$\begin{aligned} \rho_{ji,\,nm} &= \frac{\rho_{i,m}\sigma_i\,\sigma_m - \rho_{n,i}\sigma_n\,\sigma_i - \rho_{j,m}\sigma_j\,\sigma_m + \rho_{j,\,n}\sigma_j\,\sigma_n}{\sqrt{\sigma_j^2 + \sigma_i^2 - 2\,\rho_{j,i}\,\sigma_j\,\sigma_i}\,\sqrt{\sigma_n^2 + \sigma_m^2 - 2\,\rho_{n,\,m}\,\sigma_n\,\sigma_m}} \\ &= \frac{\rho_{i,m}\sigma_i\,\sigma_m - \rho_{n,i}\sigma_n\,\sigma_i - \rho_{j,m}\sigma_j\,\sigma_m + \rho_{j,\,n}\sigma_i\,\sigma_m}{\sigma_{ji}\,\sigma_{nm}} \end{aligned} \tag{4.8}$$

  - The correlation of two cross rates with respect to the same base currency is:

$$\begin{aligned} \rho_{jb,\,nb} &= \frac{\sigma_b^2 - (\rho_{n,b}\sigma_n + \rho_{j,b}\sigma_j)\,\sigma_b + \rho_{j,\,n}\sigma_j\,\sigma_n}{\sqrt{\sigma_j^2 + \sigma_b^2 - 2\,\rho_{j,b}\,\sigma_j\,\sigma_b}\,\sqrt{\sigma_n^2 + \sigma_b^2 - 2\,\rho_{n,\,b}\,\sigma_n\,\sigma_b}} \\ &= \frac{\sigma_b^2 - (\rho_{n,b}\sigma_n + \rho_{j,b}\sigma_j)\,\sigma_b + \rho_{j,\,n}\sigma_j\,\sigma_n}{\sigma_{jb}\,\sigma_{nb}} \end{aligned} \tag{4.9}$$

- From Equation 4.2 (with the substitution $D = 1/D_b$, $S_1 = D_j$ and $S_2 = S$), the correlation of a cross rate D_{jb} with a risk factor S still quoted in its original foreign currency is:

$$\rho_{jb,\,S} = \frac{\rho_{j,\,S}\,\sigma_j - \rho_{b,\,S}\,\sigma_b}{\sqrt{\sigma_b^2 + \sigma_j^2 - 2\,\rho_{b,\,S}\,\sigma_b\,\sigma_S}} = \frac{\rho_{j,\,S}\,\sigma_j - \rho_{b,\,S}\,\sigma_b}{\sigma_{jb}} \tag{4.10}$$

- From Equation 4.1 (with the substitution $D = D_{jb}$), the volatility of a *transformed* risk factor S is

$$\begin{aligned}\sigma_{D_{jb}S} &= \sqrt{\sigma_{jb}^2 + \,\sigma_S^2 + 2\rho_{jb,S}\,\sigma_{jb}\,\sigma_S} \\ &= \sqrt{\sigma_{jb}^2 + \,\sigma_S^2 + 2\,(\sigma_j\rho_{j,\,S}\,\sigma_S - \sigma_b\rho_{b,\,S}\sigma_S\,)}\end{aligned} \tag{4.11}$$

  – with $\rho_{jb,\,S}$ from Equation 4.10.

- The correlation of a risk factor S quoted in the investor's home currency (by applying the corresponding (cross) exchange rate D_{jb}) with a cross rate D_{nb} results from Equation 4.2 with the substitutions $D = D_{jb}$, $S_1 = S$ and $S_2 = D_{nb}$:

$$\rho_{D_{jb}S,\,D_{nb}} = \frac{\rho_{jb,\,nb}\,\sigma_{jb} \,+ \rho_{nb,\,S}\,\sigma_S}{\sqrt{\sigma_{jb}^2 + \sigma_S^2 + 2\,\rho_{jb,\,S}\,\sigma_{jb}\,\sigma_S}} \tag{4.12}$$

- With:
  * $\rho_{jb,\,nb}$ as in Equation 4.9
  * $\rho_{jb,\,S}$ as in Equation 4.10
  * σ_{jb} as in Equation 4.7.
- This can of course also be derived from first principles (with $\sigma_{D_{jb}S}$ we use Equation 4.11):

$$\begin{aligned}
&\operatorname{cov}\left[\delta \ln\left(D_{jb}S\right),\ \delta \ln\left(D_{nb}\right)\right] \\
&= \operatorname{cov}\left[\delta \ln\left(D_{jb}\right) + \delta \ln\left(S\right),\ \delta \ln\left(D_{nb}\right)\right] \\
&= \operatorname{cov}\left[\delta \ln\left(D_{jb}\right), \delta \ln\left(D_{nb}\right)\right] + \operatorname{cov}\left[\delta \ln\left(S\right),\ \delta \ln\left(D_{nb}\right)\right]
\end{aligned}$$

$$\rho_{D_{jb}S,\,D_{nb}} \sigma_{D_{jb}S}\ \sigma_{nb} = \rho_{jb,\,nb}\sigma_{jb}\sigma_{nb} + \rho_{nb,\,S}\,\sigma_S\sigma_{nb}$$

- If both cross rates are the same:

$$\rho_{D_{jb}S,\,D_{jb}} = \frac{\sigma_{jb}\ + \rho_{jb,S}\,\sigma_S}{\sqrt{\sigma_{jb}^2 + \sigma_S^2 + 2\,\rho_{jb,\,S}\,\sigma_{jb}\,\sigma_S}} \tag{4.13}$$

- The correlation of a risk factor S_1 (quoted in the base currency by applying cross exchange rate D_{jb}) with another risk factor S_2 (quoted in the base currency by applying cross exchange rate D_{ib}) results from Equation 4.4 with the substitutions $D_1 = D_{jb}$ and $D_2 = D_{ib}$:

$$\rho_{D_{jb}S_1,\, D_{ib}S_2} = \frac{\rho_{jb,\,ib}\sigma_{jb}\,\sigma_{ib} + \rho_{jb,\,S_2}\sigma_{jb}\,\sigma_{S_2} + \rho_{ib,\,S_1}\sigma_{ib}\,\sigma_{S_1} + \rho_{S_1,\,S_2}\sigma_{S_1}\,\sigma_{S_2}}{\sqrt{\sigma_{jb}^2 + \sigma_{S_1}^2 + 2\,\rho_{jb,\,S_1}\,\sigma_{jb}\,\sigma_{S_1}}\;\sqrt{\sigma_{ib}^2 + \sigma_{S_2}^2 + 2\,\rho_{ib,\,S_2}\,\sigma_{ib}\,\sigma_{S_2}}} \tag{4.14}$$

  - with:
    * $\rho_{jb,\,ib}$ as in Equation 4.9
    * the correlations between cross rates and risk factors (like for instance like $\rho_{jb,\,S_2}$, $\rho_{ib,\,S_1}$, etc.) as in Equation 4.10
    * the cross rate volatilities σ_{jb} and σ_{ib} as in Equation 4.7
    * the correlation $\rho_{S_1,\,S_2}$ and volatilities σ_{S_1} and σ_{S_2} as they come from the data provider
  - This can of course also be derived from first principles (with Equation 4.11

for $\sigma_{D_{jb}S_1}$ and $\sigma_{D_{ib}S_2}$):

$$\begin{aligned}
&\operatorname{cov}\left[\delta \ln\left(D_{jb}S_1\right),\ \delta \ln\left(D_{ib}S_2\right)\right] \\
&= \operatorname{cov}\left[\delta \ln\left(D_{jb}\right) + \delta \ln\left(S_1\right),\ \delta \ln\left(D_{ib}\right) + \delta \ln\left(S_2\right)\right] \\
&= \operatorname{cov}\left[\delta \ln\left(D_{jb}\right), \delta \ln\left(D_{ib}\right)\right] + \operatorname{cov}\left[\delta \ln\left(D_{ib}\right), \delta \ln\left(S_1\right)\right] \\
&+ \operatorname{cov}\left[\delta \ln\left(D_{jb}\right), \delta \ln\left(S_2\right)\right] + \operatorname{cov}\left[\delta \ln\left(S_1\right), \delta \ln\left(S_2\right)\right]
\end{aligned}$$

$$\rho_{D_{jb}S_1,\, D_{ib}S_2}\sigma_{D_{jb}S_1}\ \sigma_{D_{ib}S_2} = \rho_{jb,\, ib}\sigma_{jb}\sigma_{ib} + \rho_{ib,\, S_1}\sigma_{ib}\sigma_{S_1} + \rho_{jb,\, S_2}\sigma_{jb}\sigma_{S_2}$$

– If both cross rates are the same:

$$\rho_{D_{jb}S_1,\, D_{jb}S_2} = \frac{\sigma_{jb}^2 + \left(\rho_{jb,\, S_2}\sigma_{S_2} + \rho_{jb,\, S_1}\sigma_{S_1}\right)\sigma_{jb} + \rho_{S_1,\, S_2}\sigma_{S_1}\ \sigma_{S_2}}{\sqrt{\sigma_{jb}^2 + \sigma_{S_1}^2 + 2\,\rho_{jb,\, S_1}\,\sigma_{jb}\,\sigma_{S_1}}\ \sqrt{\sigma_{jb}^2 + \sigma_{S_2}^2 + 2\,\rho_{jb,\, S_2}\,\sigma_{ib}\,\sigma_{S_2}}} \tag{4.15}$$

5

Example of a VaR Computation

- We now explicitly compute the VaR of a concrete portfolio within the delta-normal method.
- All of the concepts introduced above are needed in the calculation.
- The example is a complete reference containing all the essential steps for a VaR computation in the “simple” delta-normal approximation.

- We will proceed step by step through the example.

5.1 The Portfolio

The portfolio is presented in Figure 5.1. It is composed of

- A British zero bond (denoted by GBP.R180) with a lifetime of 6 months and a face value of 100,000 British pounds.
- A Japanese zero bond (denoted by JPY.Z06) with a lifetime of 6 years and a face value of 2,000,000 Japanese yen.
- Put options on Japanese 7-year zero bonds (denoted by JPY.Z07) with a face value of 6,000,000 Japanese yen.
  - Lifetime of 6 months
  - Strike of 94% (of the face value)
  - The Black-Scholes price is 0.389% (of the face value), i.e. 23,364 JPY.
  - The delta is -0.2803.

Financial Instrument	Currency	Market Price	Cross Rate to EUR	Principal in Currency	Market Value in EUR
GBP.R180	GBP	97.70%	1.61720 €	£10,000.00	15,800.68 €
JPY.Z06	JPY	96.01%	0.00926 €	2,000,000 JPY	17,777.28 €
Put on JPY.Z07	JPY	0.389%	0.00926 €	6,000,000 JPY	216.30 €
Delta of Put		-0.2803		Sum	33,794.27 €

Figure 5.1: The portfolio.

5.2 Market Data

5.2.1 The Given Market Data

- For GPG.R180, JPY.Z07 and JPY.Z05we are given (see Figure 5.2):
  - Prices
  - Daily volatilities (both price as well as the *yield* volatilities)
  - Correlations are presented in Figure 5.3.

5.2.2 Filling the Gaps

- Data for a 6-year zero bond in Japanese yen (JPY.Z06) is not available.
  - Its yield and its price volatility are deduced from the vertex data at 5 and 7 years
    - Linear interpolation as in Equations 3.2 and 3.3.
  - The zero bond price is then calculated from its yield with the annual compounding methode.
  - Its yield volatility is then determined using Equation 3.1.
  - The data in bold-faced italics in Figure 5.2 are therefore not original market values, but have been obtained through interpolation and other calculations.

5.2.3 Rebasing the Market Data and the Statistical Data

- The *reference currency* for the portfolio is euro (EUR).

Risk Factor	Currency	Price		Yield	
		Market Price	Daily Price Vol	Annual Yield	Daily Yield Vol
USD/EUR	USD	$0.9083	0.680%		
USD/GBP	USD	$1.4689	0.404%		
USD/JPY	USD	$0.0084	0.731%		
GBP.R180	GBP	97.70%	0.026%	4.70%	1.10%
JPY.Z05	JPY	97.74%	0.093%	0.46%	4.01%
JPY.Z07	JPY	93.89%	0.177%	0.90%	2.99%
JPY.Z06	JPY	***96.01%***	***0.135%***	***0.68%***	***3.33%***

Figure 5.2: Market data of the portfolio's risk factors given in their original currency.

  - Therefore the FX rate of GBP andJPY with respect toEUR are also risk factors.
  - These are also listed in Figure 5.2.

- The given volatilities and correlations are with respect to their respective original currencies (GBP and JPY).
- They must be rebased to the reference currency.
  - For this, the volatilities and correlations of the FX rates are also needed.
  - However, the FX rates *themselves* are not available in EUR, but in yet *another* currency (US dollar, see Figure 5.2). Therefore:
    * The USD (with *its* volatilities and correlations) also comes into play.
    * The USD/EUR exchange rate appears in Figure 5.2 as well.

The Given Statistical Data

- The original data from a data provider is shown in Figure 5.3.

Price Volatilities and Correlations in Original Currency						
	USD/EUR	USD/JPY	USD/GBP	JPY.Z05	JPY.Z07	GBP.R180
USD/EUR	**0.680%**	0.49105	0.76692	0.13130	0.03701	-0.01863
USD/JPY	0.49105	**0.731%**	0.51365	-0.02040	-0.02092	-0.07130
USD/GBP	0.76692	0.51365	**0.404%**	0.12458	0.10580	-0.11574
JPY.Z05	0.13130	-0.02040	0.12458	**0.093%**	0.89410	0.17171
JPY.Z07	0.03701	-0.02092	0.10580	0.89410	**0.177%**	0.00987
GBP.R180	-0.01863	-0.07130	-0.11574	0.17171	0.00987	**0.026%**

Figure 5.3: The original price volatilities and correlations. The data are for relative daily changes.

- The transformations of these data into EUR-based data proceeds in four steps:

1. The volatilities and correlations of the cross rates with each other w.r.t. EUR are determined in accordance with Equations 4.7 and 4.9.

2. Now the correlations $\rho_{jb,S}$ between these cross rates and the risk factors are determined using Equation 4.10.

    - here, the risk factors are still quoted in there original currencies

3. Now we apply Equations 4.12 and 4.13 to determine the correlations between the risk factors (now quoted in the base currency) and the cross rates (also quoted in the base currency).

4. Finally (with the risk factors quoted in the base currency)

    - the EUR-based volatilities of the risk factors are calculated via Equation 4.11
    - the correlations of the risk factors with one another are determined via Equation 4.14 (or Equation 4.15 for the correlation between the two Japanese bonds).

- For Steps 3 and 4 we need
  - the EUR-based correlations and volatilities of the cross rates from Step 1
  - the correlations between the risk factors (still in their original currencies) and the cross rates obtained in Step 2
- The results are presented in Figure 5.4.
  - very high correlation between the Japanese bonds and the EUR/JPY exchange rate.
  - in contrast, the *original* correlations of those bonds with the exchange rates in Figure 5.3 which are very low.
  - the currency transformations correctly capture the fact, that those bond prices, if seen from outside Japan, are very much dependent on the JPY exchange rate.
- The statistical data for the three bonds (the portfolios of the original risk factors) are compared in Figure 5.5.

Price Volatilities and Correlations in EUR					
	EUR/JPY	EUR/GBP	JPY.Z05	JPY.Z07	GBP.R180
EUR/JPY	**0,713%**	0,49406	0,99141	0,96976	0,49221
EUR/GBP	0,49406	**0,452%**	0,48798	0,49552	0,99834
JPY.Z05	0,99141	0,48798	**0,705%**	0,99006	0,48739
JPY.Z07	0,96976	0,49552	0,99006	**0,725%**	0,49386
GBP.R180	0,49221	0,99834	0,48739	0,49386	**0,450%**

Figure 5.4: The volatilities and correlations with respect to EUR.

Volatilities and Correlations of Risk Factors						
in Original Currency				in EUR		
JPY.Z05	JPY.Z07	GBP.R180		JPY.Z05	JPY.Z07	GBP.R180
0,093%	0,89410	0,17171	JPY.Z05	0,705%	0,99006	0,48739
0,89410	0,177%	0,00987	JPY.Z07	0,99006	0,725%	0,49386
0,17171	0,00987	0,026%	GBP.R180	0,48739	0,49386	0,450%

Figure 5.5: Volatilities and correlations of the risk factors.

Note on the Side

- Note that the original data refer to *different* currencies:
  - the exchange rates all refer to the USD,
  - the Japanese bond prices to JPY
  - and the British bond price to GBP.
- Thus, even if the base currency was USD, the volatilities and correlations of the Japanese and British bonds must be converted.
  - Use Equation 4.1 for the volatilities
  - Use Equations 4.2 (with S_2 being an exchange rate) and 4.3 for the correlations between the bonds and the exchange rates
  - Use Equations 4.4 and Equation 4.5 for correlations of the bonds with one another.
  - The corresponding results for a US-based investor are presented in Figure 5.6.

Price Volatilities and Correlations in USD						
	USD/EUR	USD/JPY	USD/GBP	JPY.Z05	JPY.Z07	GBP.R180
USD/EUR	**0.680%**	0.49105	0.76692	0.50501	0.48829	0.76987
USD/JPY	0.49105	**0.731%**	0.51365	0.99192	0.97158	0.51183
USD/GBP	0.76692	0.51365	**0.404%**	0.52664	0.52665	0.99794
JPY.Z05	0.50501	0.99192	0.52664	**0.735%**	0.99058	0.52632
JPY.Z07	0.48829	0.97158	0.52665	0.99058	**0.748%**	0.52515
GBP.R180	0.76987	0.51183	0.99794	0.52632	0.52515	**0.402%**

Figure 5.6: The volatilities and correlations with respect to USD.

5.3 Cash Flow Mapping

- Now that all required data is at our disposal, we can proceed with the actual computation of the value at risk.
- We begin by distributing the face value of the 6-year zero bond among the JPY.Z07 and JPY.Z05 vertices.
  - We use risk-based cash flow mapping, Equations 3.8 and 3.9.

Risk-Based Cash Flow Mapping			
Root	1.175E-06	a	0.4539195
α_1	0.4621096	b	0.5500128
α_2	2.69074772	α_2 irrelevant, as > 1	

Figure 5.7: Calculation of the weights needed for the cash flow mapping.

- The mapping occurs entirely within the the original currency of the bonds (JPY).
  - Thus, the volatilities and correlations of JPY.Z07 and JPY.Z05 in the *original* currency are used.
- The cash flow mapping is presented in Figure 5.7.
  - 45.4 % of the face value of the 6-year bond is transferred to the 5-year vertex
  - 55.0% of the face value of the 6-year bond is transferred to the 7-year vertex.
  - The weights do not add up to 100% because $\alpha + \beta = 1$, not $a + b$ (see 3.7).

Financial Instrument	Currency	Market Price	Cross Rate to EUR	Principal in Currency	Market Value in EUR
GBP.R180	GBP	97.70%	1.61720 €	£10,000.00	15,800.68 €
JPY.Z05	JPY	97.74%	0.00926 €	907,839 JPY	8,215.05 €
JPY.Z07	JPY	93.89%	0.00926 €	1,100,026 JPY	9,562.23 €
Put on JPY.Z07	JPY	0.389%	0.00926 €	6,000,000 JPY	216.30 €
Delta of Put		-0.2803		Sum	33,794.27 €

Figure 5.8: The portfolio after the cash flow mapping.

- The portfolio after the cash flow mapping is presented in Figure 5.8.

5.4 Calculation of Risk

- The risk of the portfolio is to be calculated in EUR.
  - The rebased volatilities and correlations in Figure 5.5 are needed.

Value at Risk			
Days 10		Confidence 2,326	
Risk Factor	VaR of Each Position in EUR	Portfolio Delta in EUR with Respect to Each Risk Factor	Portfolio VaR in EUR with Respect to Each Risk Factor
GBP.R180	523,45 €	16.171,97 €	523,45 €
JPY.Z05	426,28 €	8.404,73 €	426,28 €
JPY.Z07	509,86 €	- 5.383,26 €	269,51 €
Put on JPY.Z07	779,37 €	Sum	1.219,25 €
Sum	2.238,96 €	Total VaR	615,66 €

Figure 5.9: Three ways for calculating the value at risk.

- We calculate three different risk values in Figure 5.9 to illustrate
  - the *hedging effects* (the position in the 7-year bond partially hedges the put on the 7-year bond),
  - and the *correlation/diversification effects.*
- Firstly, the value at risk is calculated for each separate *position* with Equation

2.7 (drifts are all neglected in the entire example).

  - If neither hedging nor diversification effects are to be taken into account, we could simply add the four values at risk of the separate positions and arrive at a total (very conservative) risk number.

- Secondly, the value at risk of the entire portfolio with respect to each separate *risk factor* is determined, likewise with Equation 2.7. This takes the hedge effects into account.

  - First calculate the portfolio deltas with respect to the three risk factors.
  - The deltas of the bond positions are just the face values.
    - Because the discount factors (and not the yields) are directly the risk factors.
  - The delta of the option position is the option delta multiplied by the face value of the option.
  - The deltas with respect to the same risk factor can simply be added to yield the portfolio delta with respect to that risk factor.

        - The hedge effect of the put is in this way taken into consideration.
        - The delta of the put on the 7-year bond is negative and substantially reduces the portfolio sensitivity with respect to the 7-year bond.
            - It more than compensates the positiv delta of the 7-year bond position since the resulting portfolio delta with respect to the risk factor JPY.Z07 is negativ.
            - This is called *over hedging*.
    - We now add the three portfolio VaRs with respect to each risk factor to arrive at a total (still conservative) risk number.
    - The hedge effect of the put position is considerable.
    - The put reduces the value at risk of the portfolio to almost half of the original conservative figure.

- Finally, the value at risk is computed taking both hedge and correlation / diversification effects into account,
    - via Equation 2.9

  - using the correlations with respect to the reference currency EUR listed in Figure 5.5.

- The correlation (respectively diversification) effects reduce the value at risk again by roughly one half.

- Since JPY.Z07 is strongly *correlated* to JPY.Z05, the negative portfolio delta w.r.t. JPY.Z07significantly reduces the *total* risk of the portfolio, when correlation effects are taken into account.

- Assuming that all model assumptions and approximations made are justified, we can now be 99% confident that the portfolio will lose no more than this Value at Risk over the next ten days.

6

Essential Statistical Tools

6.0.1 Moment Generating Functions

- The *moment generating function* (in short *MGF*) of a random variable x with density function $\mathrm{pdf}(x)$ is defined as the expectation of e^{sx} for an arbitrary real value s

$$G_x(s) = \mathrm{E}[e^{sx}] = \int_{-\infty}^{\infty} e^{sx}\,\mathrm{pdf}(x)dx \tag{6.1}$$

- This corresponds to the *Laplace transformation* of the pdf.

- TheMGF has very useful property: If two random variables x and y are *independent* then

$$G_{x+y}(s) = G_x(s)\, G_y(s) \tag{6.2}$$

  - The *distribution* of a *sum* of random variables is generally very difficult to determine (as we will see below).
  - The *MGF* of such a sum, in contrast, is simply the product of the MGFs of each of the *individual* distributions!
  - The random variables in the sum can be governed by completely *different* distributions, as long as theey are all statistical *independent.*
  - Equation 6.2 is quite simple to prove (independence is needed in the second to last step):

$$G_{x+y}(s) \equiv \mathrm{E}[e^{s(x+s)}] = \mathrm{E}[e^{sx}e^{sy}] = \mathrm{E}[e^{sx}]\mathrm{E}[e^{sy}] = G_x(s)\, G_Y(s)$$

- Similarly, for all non-stochastic values a, b and random variables x we have

$$G_{ax+b}(s) = e^{sb} G_x(as) \tag{6.3}$$

  - The proof is also quite simple:

$$G_{ax+b}(s) = \mathrm{E}[e^{s(ax+b)}] = e^{bs}\mathrm{E}[e^{(as)x}] = e^{sb} G_x(as)$$

- The most famous property (which gave the function its name), however, is that differentiating the MGF with respect to s at $s = 0$ yields all moments of the distribution:

$$\left. \frac{\partial^n G_x(s)}{\partial s^n} \right|_{s=0} = \mathrm{E}[x^n] \tag{6.4}$$

  - This can be shown by Taylor-expanding e^{sx} and then differentiating with respect to s (**Assignment**).

- The *central* moments of a random number x with expectation $\mu = \mathrm{E}[x]$ are the moments of the random number $\widetilde{x} := x - \mu$.

- Equation 6.3 yields for the moments of $\widetilde{x}$

$$G_{\widetilde{x}}(s) = G_{x-\mu}(s) = e^{-s\mu} G_x(s) \tag{6.5}$$

- Thus, the central moments of x can also be claculated directly from the MGF of x :

$$\mathrm{E}[(x - \mathrm{E}[x])^n] = \frac{\partial^n}{\partial s^n} \exp\left(-s\mathrm{E}[x]\right) G_x(s)\bigg|_{s=0} \tag{6.6}$$

- The general procedure for calculating central moments is therefore:
  * first calculate the expectation using Equation 6.4.
  * Then insert the result into Equation 6.6 for the central moments.

- For many distributions an *explicit analytical expression* for the MGF can be obtained from the integral representation 6.1.

- The moments can then be calculated by simply differentiating this analytical expression.

Example: The Normal Distribution

- The MGF of the standard normal distribution is, by Definition 6.1

$$
\begin{aligned}
G_{\mathrm{N}(0,1)}(s) = \mathrm{E}[e^{sx}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 - 2sx}{2}\right\} dx \\
&= \exp\left(\frac{1}{2}s^2\right) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-s)^2}{2}\right\} dx}_{1}
\end{aligned}
$$

  - We completed the square in the last step.
  - The remaining integral is the probability that a $\mathrm{N}(s, 1)$-distributed random variable will take on *any* value.

- The moment generating function of the *standard* normal distribution is thus simply

$$
G_{\mathrm{N}(0,1)}(s) = \exp\left(\frac{1}{2}s^2\right) \tag{6.7}
$$

- The moment generating function of the normal distribution with expectation μ and variance σ now follows from 6.3

$$G_{\mathrm{N}(\mu,\sigma^2)}(s) = e^{\mu s} G_{\mathrm{N}(0,1)}(\sigma s) = \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right) \tag{6.8}$$

  - From this explicit analytical expression, all *moments* can be calculated using Equation 6.4.

- The MGF for the *central* moments is even simpler in this case:

$$\begin{aligned} \mathrm{E}[(x - \mathrm{E}[x])^n] &= \frac{\partial^n}{\partial s^n} \exp(-s\mu) \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right)\bigg|_{s=0} \\ &= \frac{\partial^n}{\partial s^n} \exp\left(\frac{1}{2}\sigma^2 s^2\right)\bigg|_{s=0} \end{aligned}$$

– The first few moments are (**Assigment**) explicitly:

$$\begin{aligned} \mathrm{E}[x] &= \mu \\ \mathrm{E}[(x-\mathrm{E}[x])^2] &= \sigma^2 \\ \mathrm{E}[(x-\mathrm{E}[x])^4] &= 3\sigma^4 \\ \mathrm{E}[(x-\mathrm{E}[x])^n] &= 0 \quad \text{for all odd } n > 2 \end{aligned} \tag{6.9}$$

- From these moments the *skewness* and the *curtosis* of the normal distribution follow directly:

$$\begin{aligned} \text{Skewness} := \frac{\mathrm{E}[(x-\mathrm{E}[x])^3]}{\mathrm{E}[(x-\mathrm{E}[x])^2]^{3/2}} &= 0 \\ \text{Curtosis} := \frac{\mathrm{E}[(x-\mathrm{E}[x])^4]}{\mathrm{E}[(x-\mathrm{E}[x])^2]^2} &= 3 \end{aligned} \tag{6.10}$$

6.0.2 Characteristic Functions

- Similar MGF, the *characteristic function* (in short CF)of a random variable x is defined as the expectation of e^{isx}:

$$\Phi_x(s) := \mathrm{E}[e^{isx}] = \int_{-\infty}^{\infty} e^{isx}\,\mathrm{pdf}(x)dx \tag{6.11}$$

  - Here i denotes the imaginary number satisfying $i^2 = -1$.
  - The CF is just the *Fourier transformation* of the pdf .

- The advantage of the CF is that its inverse, the *inverse Fourier transformation* always exists:

$$\mathrm{pdf}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\Phi_x(s)ds \tag{6.12}$$

- The validity of Equation 6.12 can be shown quite easily:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\Phi_x(s)ds &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\int_{-\infty}^{\infty} e^{isx'}\,\mathrm{pdf}(x')dx'ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-is(x-x')}\,\mathrm{pdf}(x')dx'ds \\
&= \int_{-\infty}^{\infty} \delta(x-x')\,\mathrm{pdf}(x')dx' \\
&= \mathrm{pdf}(x)
\end{aligned}$$

  - where the *Dirac delta function* was used in the above derivation:

$$\delta(x-x^{'}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(x-x^{'})}ds$$

  - This is not a function but a *distribution* with the defining property

$$\int_{-\infty}^{\infty} \delta(x-x')f(x')dx' := f(x)$$

- Precisely this property yields the invertibility of the Fourier transformation.

- Thus, if Φ_x is known, then the *distribution* can be computed *directly* (and not only its moments as was the case with the MGF).
- Analogously to the MGF, Equation 6.2 holds for the characteristic function as well, i.e. for independent random variables x and y:

$$\Phi_{x+y}(s) = \Phi_x(s)\,\Phi_y(s) \tag{6.13}$$

- Likewise, for non-stochastic values a, b and a random variable x

$$\Phi_{ax+b}(s) = e^{ibs}\Phi_x(as) \tag{6.14}$$

- CF can (usually) be obtained by simply substituting is for s in the corresponding MGF.
  - For instance, the CF for the normal distribution is:

$$\Phi_{\mathrm{N}(\mu,\sigma^2)}(s) = e^{i\mu s}\Phi_{\mathrm{N}(0,1)}(\sigma s) = \exp\left(i\mu s - \frac{1}{2}\sigma^2 s^2\right) \tag{6.15}$$

6.0.3 The χ^2-Distribution

- A sum of normally distributed random variables is itself normally distributed.
- In the Delta-Gamma method we will also need to take sums of the *squares* of random variables.
- The sum of n *squared, independent, standard normally distributed random variables,* $x_i, (i = 1, \ldots, n)$ has a distribution known as the χ^2*-distribution with* n *degrees of freedom*

$$x_i \sim \mathrm{N}(0,1) \ , \quad i = 1, \ldots, n \ , \quad x_i \ \ iid \implies \sum_{i=1}^{n} x_i^2 =: y \sim \chi^2(n) \tag{6.16}$$

- Motivation for the name "degree of freedom":
  - A $\chi^2(n)$-distributed variable can be thought of as being "made up" of n *independent* (standard normal) random variables.
- We only need the case $n = 1$

- The *square* of a standard normal random variable x is governed by the χ^2*-distribution with one degree of freedom.*

$$x \sim \mathrm{N}(0,1) \Longrightarrow x^2 =: y \sim \chi^2(1) \tag{6.17}$$

- The moment generating function of $\chi^2(1)$ is

$$G_{\chi^2(1)}(s) = \mathrm{E}[e^{sx^2}]_{\mathrm{N}(0,1)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx^2} e^{-x^2/2} dx \tag{6.18}$$

  - This integral can be solved analytically (**Assigment**):

$$G_{\chi^2(1)}(s) = \frac{1}{\sqrt{1-2s}} \tag{6.19}$$

- Since the random variables in Definition 6.16 are all *independent*, Equation 6.2 directly gives the MGF for a χ^2-distribution with n degrees of freedom

$$G_{\chi^2(n)}(s) = \frac{1}{(1-2s)^{n/2}} \quad \text{for} \quad n = 1, 2, \ldots \tag{6.20}$$

- From this, the *moments* can be derived by differentiating with respect to s (**Assignment**):

$$\mathrm{E}[x^k]_{\chi^2(n)} = \prod_{i=0}^{k-1}(n+2i) \tag{6.21}$$

  - For example, the expectation and variance are given by

$$\mathrm{E}[x]_{\chi^2(n)} = n \ , \quad \mathrm{var}\,[x]_{\chi^2(n)} = 2n$$

- An explicit expression for the density function of $\chi^2(1)$ is derived in [12]:

$$\mathrm{pdf}_{\chi^2(1)}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2} \text{ with } \ x \in [0, \infty[\tag{6.22}$$

- It it also shown in [12] that the $\chi^2(n)$ equals the gamma distribution with parameters $\lambda = 1/2$ and $t = n/2$:

$$\mathrm{pdf}_{\chi^2(n)}(x) = \frac{1}{\Gamma(n/2)} \left(\frac{1}{2}\right)^{n/2} x^{\frac{n}{2}-1} e^{-x/2} \text{ with } \ x \in [0, \infty[\ , \ \ n = 1, 2, \ldots \tag{6.23}$$

– The gamma functions appearing here are:

$$\Gamma(n/2) = \begin{cases} (n/2-1)! & \text{for even values of } n \\ (n/2-1)(n/2-2)(n/2-3)\cdots(1/2)\sqrt{\pi} & \text{for odd values of } n \end{cases}$$

- The characteristic function of the χ^2-distribution is obtained by replacing s with is in the MGF:

$$\Phi_{\chi^2(n)}(s) = \frac{1}{(1-2is)^{n/2}} \quad \text{for} \quad n = 1, 2, \ldots \tag{6.24}$$

The Non-Central χ^2-Distribution

- The χ^2-distribution described above is the distribution of a sum of n squared independent *standard* normal random numbers x_i, $(i = 1, \ldots, n)$.
- Now: a slight but often needed generalization: the x_i have expectations $\mu_i \neq 0$.
- The distribution of a sum of n squared random numbers of this type is called the *non-central* χ^2-distribution with n degrees of freedom and with *non-central*

parameter θ, where θ denotes the sum of the squared expectations μ_i:

$$x_i \sim \mathrm{N}(\mu_i, 1) \quad , \quad i = 1, \ldots, n \quad , \quad x_i \;\; iid$$
$$\Longrightarrow$$
$$\sum_{i=1}^{n} x_i^2 =: y \sim \chi^2(n, \theta) \quad \text{with} \quad \theta = \sum_{i=1}^{n} \mu_i^2 \tag{6.25}$$

- The square of a single random number x $\sim$N$(0, \mu)$ has the non-central χ^2-distribution with one degree of freedom:

$$x \sim \mathrm{N}(\mu, 1) \Longrightarrow x^2 =: y \sim \chi^2(1, \mu^2)$$

- To determine the MGF of the non-central χ^2-distribution we again start from Equation **??**:

$$\begin{aligned} G_{\chi^2(1,\mu^2)}(s) &= \mathrm{E}[e^{sx^2}]_{\mathrm{N}(\mu,1)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx^2} e^{-(x-\mu)^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{ sx^2 - \frac{1}{2}(x-\mu)^2 \right\} dx \end{aligned}$$

- By completing the square in the exp-function this integral can be solved analytically (**Assigment**) to yield:

$$G_{\chi^2(1,\mu^2)}(s) = \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{s\mu^2}{1-2s}\right\} \tag{6.26}$$

- Equation 6.2 now immediately yields the MGF for a non-central χ^2-distribution with n degrees of freedom:

$$G_{\chi^2(n,\theta)}(s) = \frac{1}{(1-2s)^{n/2}} \exp\left\{\frac{s\,\theta}{1-2s}\right\} \quad \text{with} \quad \theta = \sum_{j=1}^{n} \mu_j^2 \tag{6.27}$$

- The characteristic function, Equation 6.11, of the non-central χ^2-distribution follows again by replacing s by is in Equation 6.27.

$$\Phi_{\chi^2(n,\theta)}(s) = \frac{1}{(1-2is)^{n/2}} \exp\left\{\frac{is\,\theta}{1-2is}\right\} \quad \text{mit} \quad \theta = \sum_{j=1}^{n} \mu_j^2 \tag{6.28}$$

7

The Delta-Gamma Method

- Taylor expansion 2.2 of the portfolio value up to second order with proxy 2.1 for the risk factors:

$$\delta V(\mathbf{S}(t)) = \widetilde{\mathbf{\Delta}}^T \delta\mathbf{Z} + \frac{1}{2}\delta\mathbf{Z}^T \widetilde{\mathbf{\Gamma}} \delta\mathbf{Z} \tag{7.1}$$

  - where

$$\delta\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \delta\mathbf{\Sigma}) \Longleftrightarrow \mathrm{cov}[\delta Z_i, \delta Z_j] = \sigma_i \rho_{ij} \sigma_j \delta t \ , \ \ \mathrm{E}\left[\delta Z_i\right] = 0$$

- This is *not* the sum over the contribution of each risk factor:

$$\sum_{i}^{n} \widetilde{\Delta}_i \, \delta Z_i + \frac{1}{2} \sum_{i,j}^{n} \delta Z_i \widetilde{\Gamma}_{ij} \, \delta Z_j \neq \sum_{i}^{n} (\text{Contribution of the } i\text{-th Risk Factor})$$

  - The contributions of the δZ_j are *coupled* by the Gamma matrix $\widetilde{\mathbf{\Gamma}}$.
  - The δZ_j themselves are *correlated* through the covariance matrix $\delta\mathbf{\Sigma}$.
  - The VaR can not be obtained from the distribution of the individual *risk factors.* The (unknown) distribution of the *portfolio value* δV itself must be determined.

- Three steps
  - Cholesky decomposition of the covariance matrix $\delta\mathbf{\Sigma}$ to transform the δZ_j into *independent* random variables.
  - Diagonalisation of the Gamma matrix $\widetilde{\mathbf{\Gamma}}$ to *decouple* the contributions of the δZ_j.
  - Determination of the *distribution* of δV.

7.1 Decoupling of the Risk Factors

- Via Cholesky decomposition a matrix $\mathbf{A}$ can be constructed satisfying the properties 1.26 and 1.30, i.e.

$$\mathbf{A}\mathbf{A}^T = \delta\mathbf{\Sigma} \Longrightarrow \mathbf{A}^{-1}\delta\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \mathbf{1})$$

- Introduce identity matrices into Equation 7.1 and replace them with $\mathbf{A}\mathbf{A}^{-1}$ or $(\mathbf{A}^T)^{-1}\mathbf{A}^T$:

$$\begin{aligned}
\delta V(\mathbf{S}(t)) &= \widetilde{\mathbf{\Delta}}^T\,\mathbf{1}\,\delta\mathbf{Z} + \frac{1}{2}\delta\mathbf{Z}^T\,\mathbf{1}\,\widetilde{\mathbf{\Gamma}}\,\mathbf{1}\,\delta\mathbf{Z} \\
&= \widetilde{\mathbf{\Delta}}^T\,\mathbf{A}\mathbf{A}^{-1}\,\delta\mathbf{Z} + \frac{1}{2}\delta\mathbf{Z}^T\,(\mathbf{A}^T)^{-1}\mathbf{A}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{A}\mathbf{A}^{\mathbf{-1}}\,\delta\mathbf{Z} \\
&= \widetilde{\mathbf{\Delta}}^T\,\mathbf{A}\mathbf{A}^{-1}\,\delta\mathbf{Z} + \frac{1}{2}\delta\mathbf{Z}^T\,(\mathbf{A}^{-1})^T\mathbf{A}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{A}\mathbf{A}^{\mathbf{-1}}\,\delta\mathbf{Z} \\
&= \widetilde{\mathbf{\Delta}}^T\mathbf{A}\,(\mathbf{A}^{-1}\,\delta\mathbf{Z}) + \frac{1}{2}(\mathbf{A}^{-1}\delta\mathbf{Z})^T\mathbf{A}^T\widetilde{\mathbf{\Gamma}}\mathbf{A}\;(\mathbf{A}^{-1}\delta\mathbf{Z})
\end{aligned}$$

- $\delta\mathbf{Z}$ only appears in combination with $\mathbf{A}^{-1}$.

– $\mathbf{A}^{-1}\delta\mathbf{Z}$ are *iid* random variables! Thus, the first goal has been accomplished:

$$\begin{aligned} \delta V(\mathbf{S}(t)) &= \widetilde{\boldsymbol{\Delta}}^T \mathbf{A}\, \delta\mathbf{Y} + \frac{1}{2} \delta\mathbf{Y}^T \mathbf{M}\, \delta\mathbf{Y} \\ \text{with } \ \delta\mathbf{Y} &:= \mathbf{A}^{-1}\delta\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, \mathbf{1}) \ , \ \ iid \\ \text{and } \ \mathbf{M} &:= \mathbf{A}^T \widetilde{\boldsymbol{\Gamma}} \mathbf{A} \end{aligned} \tag{7.2}$$

– Because $\widetilde{\boldsymbol{\Gamma}}$ is by definition a symmetric matrix, i.e. $\widetilde{\Gamma}_{ij} = \widetilde{\Gamma}_{ji}$, the newly defined matrix $\mathbf{M}$ is symmetric as well (**Assignment**).

– The δY_i are *independent, identically distributed* (*iid) standard normal* random variables.

7.2 Diagonalization of the Gamma Matrix

- Diagonalize the transformed Gamma matrix $\mathbf{M}$.

  – Standard procedure in *linear algebra* (see [25], for example).

- We demonstrate the procedure here since it entails essential elements of the practical VaR computations in the Delta-Gamma method.

- The *eigenvectors* $\mathbf{e}^i$ of a matrix $\mathbf{M}$ are the vectors which are mapped by $\mathbf{M}$ to the same vector multiplied by a number (called *scalar* in algebra):

$$\begin{aligned} \mathbf{M}\mathbf{e}^i &= \lambda_i \mathbf{e}^i \Leftrightarrow \\ (\mathbf{M} - \lambda_i \mathbf{1})\, \mathbf{e}^i &= \mathbf{0} \end{aligned} \tag{7.3}$$

- These scalars, λ_i, are called *eigenvalues* of the matrix.

- This has a non-trivial solution $(\mathbf{e}^i \neq \mathbf{0})$ iff the matrix $(\mathbf{M} - \lambda_i \mathbf{1})$ is *singular*, i.e. the *determinant* must be zero:

$$\det (\mathbf{M} - \lambda_i \mathbf{1}) = 0 \tag{7.4}$$

  - The solutions of this Equation 7.4 are the eigen*values* λ_i.
  - Once they have been determined, they can be substituted into Equation 7.3 to determine the eigen*vectors* $\mathbf{e}^i$.

  - The eigenvectors are yet only defined up to a multiplicative scalar since if $\mathbf{e}^i$ solves Equation 7.3 then $c\mathbf{e}^i$ does as well. Demand that the eigenvectors have norm 1:

$$(\mathbf{e}^i)^T\mathbf{e}^i = 1 \tag{7.5}$$

- A *symmetric* $n \times n$ matrix has n linearly independent *orthogonal* eigenvectors.
- Together with the normalization the eigenvectors are *orthonormal*:

$$(\mathbf{e}^i)^T\mathbf{e}^j = \sum_k e^i_k e^j_k = \delta_{ij} \tag{7.6}$$

- Since the matrix $\mathbf{M}$ in Equation 7.2 is symmetric, it has n orthonormal eigenvectors.

$$\mathbf{e}^j = \begin{pmatrix} e^j_1 \\ e^j_2 \\ \vdots \\ e^j_n \end{pmatrix}, \quad (\mathbf{e}^j)^T = \begin{pmatrix} e^j_1 & e^j_2 & \cdots & e^j_n \end{pmatrix}, \quad j = 1, \ldots, n$$

– Notation: e^i_k is he kth component of the ith eigenvector.

- Now construct a matrix $\mathbf{O}$ whose columns are the eigen*vectors* of $\mathbf{M}$:

$$\mathbf{O} = \begin{pmatrix} \mathbf{e}^1 & \mathbf{e}^2 & \ldots & \mathbf{e}^n \end{pmatrix} = \begin{pmatrix} e^1_1 & e^2_1 & \cdots & e^n_1 \\ e^1_2 & e^2_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ e^1_n & \cdots & \cdots & e^n_n \end{pmatrix} \Rightarrow O_{ij} = e^j_i \tag{7.7}$$

  – The jth eigenvector $\mathbf{e}^j$ is in the jth column of the matrix.
  – The ith row contains the ith components of all eigenvectors.
  – As can be immediately verified (**Assignment**)

$$\mathbf{O}^T\mathbf{O} = \mathbf{1} \tag{7.8}$$

  – from which it follows that

$$\mathbf{O}^T = \mathbf{O}^{-1} \Rightarrow \mathbf{O}\mathbf{O}^T = \mathbf{1} \tag{7.9}$$

  - Equation 7.8 characterizes *orthonormal transformations*. (graphically: rotations)

- The eigen*values* can be used to construct a diagonal matrix:

$$\lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \tag{7.10}$$

- Now Equation 7.3 can be written as (**Assignment**)

$$\mathbf{MO} = \mathbf{O}\lambda \tag{7.11}$$

- Multiplying both sides of this equation *on the left* with the matrix $\mathbf{O}^T$ yields (with Equation 7.8)

$$\mathbf{O}^T\mathbf{MO} = \mathbf{O}^T\mathbf{O}\lambda = \lambda \tag{7.12}$$

  - This the desired diagonalization of $\mathbf{M}$, since λ is a diagonal matrix:

- Now introduce the diagonalized matrix $\mathbf{O}^T\mathbf{M}\mathbf{O}$ into Equation 7.2 by inserting identity matrices and subsequently replacing them by $\mathbf{O}\mathbf{O}^\mathbf{T}$ (see Equation 7.9):

$$\begin{aligned}\delta V(\mathbf{S}(t)) &= \widetilde{\boldsymbol{\Delta}}^T \mathbf{A}\,\mathbf{1}\,\delta\mathbf{Y} + \frac{1}{2}\delta\mathbf{Y}^T\,\mathbf{1}\,\mathbf{M}\,\mathbf{1}\,\delta\mathbf{Y} \\ &= \widetilde{\boldsymbol{\Delta}}^T \mathbf{A}\mathbf{O}\mathbf{O}^\mathbf{T}\delta\mathbf{Y} + \frac{1}{2}\delta\mathbf{Y}^T\mathbf{O}\underbrace{\mathbf{O}^T\mathbf{M}\mathbf{O}}_{\lambda}\mathbf{O}^\mathbf{T}\delta\mathbf{Y} \\ &= \widetilde{\boldsymbol{\Delta}}^T \mathbf{A}\mathbf{O}\left(\mathbf{O}^\mathbf{T}\delta\mathbf{Y}\right) + \frac{1}{2}\left(\mathbf{O}^T\delta\mathbf{Y}\right)^T\lambda\left(\mathbf{O}^\mathbf{T}\,\delta\mathbf{Y}\right)\end{aligned}$$

- The parentheses emphasize that $\delta\mathbf{Y}$ appears only in combination with $\mathbf{O}^\mathbf{T}$. In consequence, we can write

$$\begin{aligned}\delta V(\mathbf{S}(t)) &= \widetilde{\boldsymbol{\Delta}}^T \mathbf{A}\mathbf{O}\,\delta\mathbf{X} + \frac{1}{2}\delta\mathbf{X}^T\lambda\delta\mathbf{X} \\ \text{with}\quad \delta\mathbf{X} &:= \mathbf{O}^\mathbf{T}\delta\mathbf{Y} = \mathbf{O}^\mathbf{T}\mathbf{A}^{-1}\delta\mathbf{Z} \\ \text{and}\quad \lambda &:= \mathbf{O}^T\mathbf{M}\mathbf{O} = \mathbf{O}^T\mathbf{A}^T\widetilde{\boldsymbol{\Gamma}}\mathbf{A}\mathbf{O}\end{aligned} \tag{7.13}$$

- The δY_i were *iid*, standard normally distributed. The covariances remain invariant under the transformation $\mathbf{O}^T$ (**Assignment**). Since matrix multiplication is a *linear* transformation the new random variables δX_i remain normally distributed, i.e.:

$$\delta\mathbf{X} \sim \mathrm{N}(\mathbf{0}, \mathbf{1}) \ , \quad iid$$

- With the "transformed sensitivity vector" $\mathbf{L} := \mathbf{O}^T \mathbf{A}^T \widetilde{\mathbf{\Delta}}$ the portfolio-change has the simple form

$$\delta V(\mathbf{S}(t)) = \mathbf{L}^T \, \delta\mathbf{X} + \frac{1}{2} \delta\mathbf{X}^T \lambda \delta\mathbf{X} \quad \text{with} \quad \delta\mathbf{X} \sim \mathrm{N}(\mathbf{0}, \mathbf{1}) \tag{7.14}$$

  - expressed component-wise

$$\delta V(\mathbf{S}(t)) = \sum_i \left[L_i \, \delta X_i + \frac{1}{2} \lambda_i \delta X_i^2 \right] = \sum_i \delta V_i$$

$$\text{with} \quad \delta V_i = L_i \, \delta X_i + \frac{1}{2} \lambda_i \delta X_i^2 \ , \quad i = 1, \ldots, n \tag{7.15}$$

- Now the value change is indeed a *decoupled* sum of the individual contributions of *iid* random variables!

- At this stage, we collect all transformations involved :

$$\delta\mathbf{X} := \mathbf{O}^T\mathbf{A}^{-1}\delta\mathbf{Z}\ ,\ \ \lambda := \mathbf{O}^T\mathbf{A}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{A}\mathbf{O}\ ,\ \ \mathbf{L} := \mathbf{O}^T\mathbf{A}^T\widetilde{\mathbf{\Delta}}$$

  - Because $\mathbf{O}^T\mathbf{A}^{-1} = \mathbf{O}^{-1}\mathbf{A}^{-1} = (\mathbf{A}\mathbf{O})^{-1}$ (see Equ. 7.9) all these transformations can represented by a single matrix

$$\mathbf{D} := \mathbf{A}\mathbf{O} \tag{7.16}$$

- With this matrix, the transformations become simply

$$\delta\mathbf{X} := \mathbf{D}^{-1}\delta\mathbf{Z}\ ,\ \ \lambda := \mathbf{D}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{D}\ ,\ \ \mathbf{L} := \mathbf{D}^T\widetilde{\mathbf{\Delta}} \tag{7.17}$$

- The matrix $\mathbf{D}$

  - *directly* diagonalizes the Gamma matrix $\widetilde{\mathbf{\Gamma}}$ (as opposed to $\mathbf{O}$, which diagonalizes the matrix $\mathbf{M}$)

– and is *also* a "square root" of the covariance matrix:

$$\mathbf{D}\mathbf{D}^T = \mathbf{A}\mathbf{O}\,(\mathbf{A}\mathbf{O})^T = \mathbf{A}\mathbf{O}\mathbf{O}^T\mathbf{A}^T = \mathbf{A}\mathbf{1}\mathbf{A}^T = \mathbf{A}\mathbf{A}^T = \delta\mathbf{\Sigma} \qquad (7.18)$$

– The matrix $\mathbf{D}$ satisfies *both* tasks, namely the decoupling of the Gamma matrix *and* the transformation of the correlated random variables into uncorrelated ones.

– With the matrix $\mathbf{D}$ the equivalence of Equations 7.14 and original 7.1 is immediate:

$$\begin{aligned}\delta V(\mathbf{S}(t)) &= \left(\mathbf{D}^T\widetilde{\mathbf{\Delta}}\right)^T \mathbf{D}^{-1}\delta\mathbf{Z} + \frac{1}{2}\left(\mathbf{D}^{-1}\delta\mathbf{Z}\right)^T \mathbf{D}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{D}{\mathbf{D}^{-1}}^{}\delta\mathbf{Z} \\ &= \widetilde{\mathbf{\Delta}}^T\underbrace{\mathbf{D}\,\mathbf{D}^{-1}}_{\mathbf{1}}\delta\mathbf{Z} + \frac{1}{2}\delta\mathbf{Z}^T\underbrace{\left(\mathbf{D}^T\right)^{-1}\mathbf{D}^T}_{\mathbf{1}}\,\widetilde{\mathbf{\Gamma}}\,\underbrace{\mathbf{D}\mathbf{D}^{-1}}_{\mathbf{1}}\delta\mathbf{Z}\end{aligned}$$

7.3 The Distribution of the Portfolio-Value Changes

- δV is *not* simply the sum of normally distributed random variables. It also includes the *square* of normally distributed random variables.

- The square of a standard normally distributed random variable is χ^2-distributed with one degree of freedom.
- This we can write 7.15 as

$$\delta V(\mathbf{S}(t)) = \sum_{i=1}^{n} L_i \, \delta X_i + \frac{1}{2} \sum_{i=1}^{n} \lambda_i \widetilde{X}_i \quad \text{where } \delta X_i \sim \mathrm{N}(0,1) \ , \quad \widetilde{X}_i \sim \chi^2(1)$$

- The $\widetilde{X}_i$ are not independent from the δX_i since obviously:

$$\widetilde{X}_i = (\delta X_i)^2 \quad \forall i$$

- We now re-write δV so that *every* term is statistically independent of *every other* term.

- δX_i is independent of every other term in δV if the corresponding eigenvalue λ_i is zero.

  - We emphasize this by the index set J which contains only the indices of non-zero eigenvalues:

$$J = \{1, \ldots, n \,|\, \lambda_j \neq 0\} \tag{7.19}$$

– With this index set we can write

$$\delta V(\mathbf{S}(t)) = \sum_{i \notin J} L_i \, \delta X_i + \sum_{j \in J} L_j \, \delta X_j + \frac{1}{2} \sum_{j \in J} \lambda_j \delta X_j^2 \qquad (7.20)$$

$$= \sum_{i \notin J} L_i \, \delta X_i + \sum_{j \in J} \left[L_j \, \delta X_j + \frac{1}{2} \lambda_j \delta X_j^2 \right]$$

- The first sum is a sum of normally distributed random variables.
  - It is again a normally distributed random variable which we denote by u_0.
  - The expectation of this random variable is

$$\mathrm{E}\,[u_0] = \mathrm{E}\left[\sum_{i \notin J} L_i \, \delta X_i \right] = \sum_{i=1}^{n} L_i \underbrace{\mathrm{E}\,[\, \delta X_i]}_{0} = 0$$

- Its variance is

$$\operatorname{var}[u_0] = \operatorname{var}\left[\sum_{i \notin J} L_i\,\delta X_i\right] = \sum_{i,j \notin J} \operatorname{cov}\left[L_i\,\delta X_i, L_j\,\delta X_j\right]$$
$$= \sum_{i,j \notin J} L_i L_j \underbrace{\operatorname{cov}\left[\,\delta X_i,\ \delta X_j\right]}_{\delta_{ij}} = \sum_{i \notin J} L_i^2$$

- Thus

$$u_0 := \sum_{i \notin J} L_i\,\delta X \sim \mathrm{N}(0, \sum_{i \notin J} L_i^2)$$

- Consider now the sums over $j \in J$ in Equation 7.20.

  - To combine the *dependent* random numbers δX_j and δX_j^2 into one single random number, we complete the square:

$$\frac{1}{2}\lambda_j \delta X_j^2 + L_j\,\delta X_j = \frac{1}{2}\lambda_j \left(\delta X_j^2 + 2\frac{L_j}{\lambda_j}\,\delta X_j\right) = \frac{1}{2}\lambda_j \left(\delta X_j + \frac{L_j}{\lambda_j}\right)^2 - \frac{L_j^2}{2\lambda_j}$$

- Since δX_j is a standard normal random variable we have

$$\delta X_j \sim \mathrm{N}(0,1) \quad \Longrightarrow \quad \delta X_j + \frac{L_j}{\lambda_j} \sim \mathrm{N}\left(\frac{L_j}{\lambda_j}, 1\right)$$

- Therefore, $u_j := (\delta X_j + L_j/\lambda_j)^2$ has a *non-central* χ^2-distribution (see 6.25) with one degree of freedom and non-central parameter L_j^2/λ_j^2:

$$\left(\delta X_j + \frac{L_j}{\lambda_j}\right)^2 =: u_j \sim \chi^2\left(1, \frac{L_j^2}{\lambda_j^2}\right) \quad \forall j \in J$$

- In summary, δV has now become a sum of
  - non-central χ^2-distributed random variables u_j
  - plus a normally distributed random variable u_0
  - plus a constant,

- and all the random variables are *independent of each other*:

$$\delta V(\mathbf{S}(t)) = u_0 + \frac{1}{2}\sum_{j\in J}\lambda_j u_j - \underbrace{\frac{1}{2}\sum_{j\in J} L_j^2/\lambda_j}_{\text{constant}} \tag{7.21}$$

$$\text{with} \quad u_0 \sim \mathrm{N}(0, \sum_{i\notin J} L_i^2) \quad , \quad u_j \sim \chi^2\left(1, (L_j/\lambda_j)^2\right) \quad , \quad j \in J$$

- The problem now consists in determining the distribution of the sum of *differently* distributed independent random variables.
  - Remember: the Value at Risk at a specified confidence c is computed from the percentiles of the distribution of δV.

7.3.1 Moments of the Portfolio-value Distribution

- We first calculate moments of the distribution of δV.
- For this we differentiate the *Moment Generating Function* (*MGF*) defined in 6.1 (see 6.4).

– The MGF of the $\mathrm{N}(0, \sum_{i \notin J} L_i^2)$ -distributed random variable u_0 is

$$G_{u_0}(s) = \exp\left(\frac{1}{2}\, s^2 \sum_{i \notin J} L_i^2\right) \tag{7.22}$$

– The MGF of a $\chi^2(1, (L_j/\lambda_j)^2)$-distributed random variables u_j are

$$G_{u_j}(s) = \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{s}{1-2s}\frac{L_j^2}{\lambda_j^2}\right\} \;,\;\; j \in J \tag{7.23}$$

  * well-defined for $s < 1/2$, which is sufficient for our needs since need it for s close to zero.

- The MGF of δV in 7.21 follows directly from the properties 6.2 and 6.3:

$$G_{\delta V}(s) = \exp\left\{-s \sum_{j \in J} \frac{L_j^2}{2\lambda_j}\right\} G_{u_0}(s) \prod_{j \in J} G_{u_j}(\frac{1}{2}\lambda_j s)$$

- Inserting the above expressions for G_{u_0} and G_{u_j} yields (after simple rearraning):

$$G_{\delta V}(s) = \exp\left(\frac{1}{2}s^2\sum_{i\notin J}L_i^2\right)\prod_{j\in J}\frac{1}{\sqrt{1-\lambda_j s}}\exp\left\{\frac{1}{2}L_j^2\frac{s^2}{1-\lambda_j s}\right\} \tag{7.24}$$

- Trick: Since $\lambda_i = 0$ for all $i \notin J$, we can re-write the first exp-function:

$$\exp\left(\frac{1}{2}s^2\sum_{i\notin J}L_i^2\right) = \prod_{i\notin J}\exp\left(\frac{1}{2}s^2L_i^2\right) = \prod_{i\notin J}\frac{1}{\sqrt{1-\lambda_i s}}\exp\left\{\frac{1}{2}L_i^2\frac{s^2}{1-\lambda_i s}\right\}$$

- Using this form in Equation 7.24 allows us to write δV very compactly as a product over *all* indexes $j = 1, \ldots, n$

$$G_{\delta V}(s) = \prod_{j=1}^{n}\frac{1}{\sqrt{1-\lambda_j s}}\exp\left\{\frac{1}{2}L_j^2\frac{s^2}{1-\lambda_j s}\right\} \tag{7.25}$$

  - This is well-defined for all $s < \min_{j\in J}\left(\frac{1}{2|\lambda_i|}\right)$, which is sufficient for our needs since we are only interested in values of s close to zero.

- Now, using 6.4, arbitrary moments of δV can be computed by differentiating 7.25.

- We first abbreviate the argument of the exp-function in 7.25 as

$$a_j := \frac{1}{2} L_j^2 \frac{s^2}{1 - \lambda_j s}$$

- For the first moment we need the first derivative.

  - Application of the *product rule* yields

$$\begin{aligned}
\mathrm{E}[\delta V] &= \left. \frac{\partial G_{\delta V}(s)}{\partial s} \right|_{s=0} \\
&= \left. \frac{\partial}{\partial s} \prod_{j=1}^{n} \frac{e^{a_j}}{\sqrt{1 - \lambda_j s}} \right|_{s=0} \\
&= \left. \sum_{j=1}^{n} \left(\frac{\partial}{\partial s} \frac{e^{a_j}}{\sqrt{1 - \lambda_j s}} \right) \prod_{k=1, k \neq j}^{n} \frac{e^{a_k}}{\sqrt{1 - \lambda_k s}} \right|_{s=0}
\end{aligned}$$

– The derivative we need to calculate is

$$\frac{\partial}{\partial s}\frac{e^{a_j}}{\sqrt{1-\lambda_j s}} = e^{a_j}\frac{\partial}{\partial s}\frac{1}{\sqrt{1-\lambda_j s}} + \frac{1}{\sqrt{1-\lambda_j s}}\frac{\partial}{\partial s}e^{a_j}$$
$$= \frac{\frac{1}{2}\lambda_j e^{a_j}}{(1-\lambda_j s)^{3/2}} + \frac{\frac{1}{2}L_j^2 e^{a_j}}{\sqrt{1-\lambda_j s}}\left(\frac{2s}{1-\lambda_j s} + \frac{\lambda_j\, s^2}{(1-\lambda_j s)^2}\right)$$
$$= \frac{1}{2}\frac{e^{a_j}}{(1-\lambda_j s)^{3/2}}\left(\lambda_j + 2L_j^2 s + \lambda_j L_j^2 \frac{s^2}{1-\lambda_j s}\right)$$

– For $s = 0$ almost all terms vanish and we are left with $\lambda_j/2$. Thus $\mathrm{E}[\delta V]$ is simply

$$\mathrm{E}[\delta V] = \sum_{j=1}^{n}\frac{1}{2}\lambda_j \prod_{k\in J, k\neq j}\frac{e^{a_k}}{\sqrt{1-\lambda_k s}}\Bigg|_{s=0} = \frac{1}{2}\sum_{j=1}^{n}\lambda_j$$

- This is by definition one half times the *trace* of the eigenvalue matrix λ. With Equations 7.17 and 7.18 this becomes:

$$\mathrm{E}[\delta V] = \frac{1}{2}\,\mathrm{tr}\,(\lambda) = \frac{1}{2}\,\mathrm{tr}\left(\mathbf{D}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{D}\right) = \frac{1}{2}\,\mathrm{tr}\left(\widetilde{\mathbf{\Gamma}}\,\mathbf{D}\mathbf{D}^T\right) = \frac{1}{2}\,\mathrm{tr}\left(\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma}\right) \qquad (7.26)$$

  - Note that the drifts of all risk factors have been neglected.
  - But still the expectation of the *portfolio* changes (the drift of the portfolio) is *not* zero because *non-linear* effects were taken into consideration.
  - The Gamma matrix gives rise to the drift of δV.

- To find out more about the distribution of δV, we proceed by computing its variance.

- The variance is the second *central* moment which can be calculated via Equation

6.6:

$$
\begin{aligned}
\mathrm{var}[\delta V] &= \mathrm{E}[(\delta V - \mathrm{E}[\delta V])^2] \\
&= \frac{\partial^2}{\partial s^2} \exp\left(-s\mathrm{E}[\delta V]\right) G_{\delta V}(s)\bigg|_{s=0} \\
&= \frac{\partial^2}{\partial s^2} \exp\left(-s\frac{1}{2}\sum_{i=1}^{n}\lambda_i\right) \prod_{j=1}^{n} \frac{\exp\left(\frac{1}{2}L_j^2\frac{s^2}{1-\lambda_j s}\right)}{\sqrt{1-\lambda_j s}}\Bigg|_{s=0} \\
&= \frac{\partial^2}{\partial s^2} \prod_{j=1}^{n} \frac{1}{\sqrt{1-\lambda_j s}} \exp\left(\frac{1}{2}L_j^2\frac{s^2}{1-\lambda_j s} - \frac{1}{2}\lambda_j s\right)\Bigg|_{s=0} \\
&= \frac{\partial^2}{\partial s^2} \prod_{j=1}^{n} a_j\Bigg|_{s=0}
\end{aligned} \tag{7.27}
$$

– with the abbreviation

$$
a_j := \frac{1}{\sqrt{1-\lambda_j s}} \exp\left(\frac{1}{2}L_j^2\frac{s^2}{1-\lambda_j s} - \frac{1}{2}\lambda_j s\right)
$$

– Doing the (tedious!) differentiations (**Assignment**) finally yields

$$\mathrm{var}[\delta V] = \sum_{j=1}^{n} \left(L_j^2 + \frac{1}{2}\lambda_j^2 \right)$$

- The sum $\sum L_j^2$ is just the square of the transformed sensitivity *vector* and $\sum \lambda_j^2$ is the trace of the square of the the *matrix* of eigenvalues, i.e.

$$\mathrm{var}[\delta V] = \mathbf{L}^T\mathbf{L} + \frac{1}{2}\,\mathrm{tr}\left(\lambda^2\right)$$

- With the transformations 7.17 and the property7.18, the variance of the portfolio's value becomes

$$\begin{aligned} \mathrm{var}[\delta V] &= \widetilde{\mathbf{\Delta}}^T\mathbf{D}\mathbf{D}^T\widetilde{\mathbf{\Delta}} + \frac{1}{2}\,\mathrm{tr}\left(\mathbf{D}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{D}\mathbf{D}^T\,\widetilde{\mathbf{\Gamma}}\,\mathbf{D}\right) \\ &= \widetilde{\mathbf{\Delta}}^T\delta\mathbf{\Sigma}\widetilde{\mathbf{\Delta}} + \frac{1}{2}\,\mathrm{tr}\left(\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma}\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma}\right) \end{aligned} \tag{7.28}$$

  – The first term resulting from the linear portfolio sensitivities $\widetilde{\mathbf{\Delta}}$ is identical to the portfolio variance in the Delta-Normal method.

- The non-linear sensitivities $\widetilde{\mathbf{\Gamma}}$ effect a correction of the linear portfolio variance.

- The variance is the second *central* moment of the random variable. The *central moments* μ_i of a random variable are defined as:

$$\mu_i := \mathrm{E}[(\delta V - \mathrm{E}[\delta V])^i] \ , \quad i > 1$$

- Analogous to the approach shown above, one can continue (**Assignment**) to calculate further central moments of δV:

$$\begin{aligned}
\mu &= \mathrm{E}[\delta V] = \frac{1}{2}\,\mathrm{tr}\,\left(\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma}\right) \\
\mu_2 &= \mathrm{E}[(\delta V - \mathrm{E}[\delta V])^2] = \widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma} \widetilde{\mathbf{\Delta}} + \frac{1}{2}\,\mathrm{tr}\,\left((\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma})^2\right) \\
\mu_3 &= \mathrm{E}[(\delta V - \mathrm{E}[\delta V])^3] = 3\widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma}\,\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma}\,\widetilde{\mathbf{\Delta}} + \mathrm{tr}\,\left((\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma})^3\right) \qquad (7.29) \\
\mu_4 &= \mathrm{E}[(\delta V - \mathrm{E}[\delta V])^4] = 12\widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma}\,(\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma})^2\,\widetilde{\mathbf{\Delta}} + 3\,\mathrm{tr}\,\left((\widetilde{\mathbf{\Gamma}}\,\delta\mathbf{\Sigma})^4\right) + 3\mu_2^2
\end{aligned}$$

  - In this way, a great deal of additional information about the distribution of δV can be generated.

– For instance *skewness* and *curtosis* of the distribution of δV are[1]

$$\text{Skewness} \equiv \frac{\mu_3}{\mu_2^{3/2}} = \frac{3\widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma}\, \widetilde{\mathbf{\Gamma}}\, \delta\mathbf{\Sigma}\, \widetilde{\mathbf{\Delta}} + \mathrm{tr}\,(\widetilde{\mathbf{\Gamma}}\, \delta\mathbf{\Sigma})^3}{\left(\widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma} \widetilde{\mathbf{\Delta}} + \frac{1}{2}\,\mathrm{tr}\,(\widetilde{\mathbf{\Gamma}}\, \delta\mathbf{\Sigma})^2\right)^{3/2}}$$

$$\text{Curtosis} \equiv \frac{\mu_4}{\mu_2^2} = \frac{12\widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma}\, (\widetilde{\mathbf{\Gamma}}\, \delta\mathbf{\Sigma})^2\, \widetilde{\mathbf{\Delta}} + 3\,\mathrm{tr}\,\left((\widetilde{\mathbf{\Gamma}}\, \delta\mathbf{\Sigma})^4\right) + 3\mu_2^2}{\left(\widetilde{\mathbf{\Delta}}^T \delta\mathbf{\Sigma} \widetilde{\mathbf{\Delta}} + \frac{1}{2}\,\mathrm{tr}\,(\widetilde{\mathbf{\Gamma}}\, \delta\mathbf{\Sigma})^2\right)^2}$$

- A *percentile*, however, is needed for the computation of the value at risk as given in Equation 1.4.

Johnson Transformation

- Computation of a percentile necessitates knowledge of the distribution *directly* and not of its moments.
- One way to proceed is to assume a particular functional form of the distribution.

[1] Recall that a normal distribution has skewness 0 and kurtosis 3, see Equation 6.10.

- Then establish a relation between the parameters of this functional form and the moments of the random variable via *moment matching*.
- Since the moments can be explicitly computed (using the MGF), the parameters of the assumed distribution can thus be determined.

- For example, one could *assume* that δV normally or lognormally distributed.
  - Then Equations 7.26 and 7.28 would determien the parameter values of the assumed distribution.

- Additional functional forms for approximating the distribution of δV were suggested by Johnson [32].
  - These *Johnson transformations* have four parameters which can be determined from the first four moments in Equation 7.29.

Cornish-Fisher Expansion

- Approximate the *percentiles* of a distribution from its *moments* and the (well known) percentiles $Q^{N(0,1)}$ of the standard normal distribution.

- First transform δV into a centered and normalized random variable $\widetilde{\delta V}$:

$$\widetilde{\delta V} := \frac{\delta V - \mathrm{E}[\delta V]}{\sqrt{\mathrm{var}[\delta V]}} = \frac{\delta V - \mu}{\sqrt{\mu_2}}$$

- Cornish-Fisher expansion (see [10], [51]) for the percentiles of the distribution of $\widetilde{\delta V}$ up to the order involving the first four moments in Equation 7.29:

$$\begin{aligned} Q^{\mathrm{cpf}_{\widetilde{\delta V}}} \approx\ & Q^{\mathrm{N}(0,1)} + \frac{1}{6}\left[\left(Q^{\mathrm{N}(0,1)}\right)^2 - 1\right] \frac{\mu_3}{\mu_2^{3/2}} \\ & + \frac{1}{24}\left[\left(Q^{\mathrm{N}(0,1)}\right)^3 - 3Q^{\mathrm{N}(0,1)}\right]\left(\frac{\mu_4}{\mu_2^2} - 3\right) \\ & - \frac{1}{36}\left[2\left(Q^{\mathrm{N}(0,1)}\right)^3 - 5Q^{\mathrm{N}(0,1)}\right]\left(\frac{\mu_3}{\mu_2^{3/2}}\right)^2 \end{aligned} \tag{7.30}$$

- The probability that $\widetilde{\delta V}$ is less than a number a is, naturally, the same as the probability that δV is less than $\mu + \sqrt{\mu_2}\, a$:

$$Q^{\mathrm{cpf}_{\delta V}} \approx \mu + \sqrt{\mu_2}\, Q^{\mathrm{cpf}_{\widetilde{\delta V}}}$$

- From Equation 1.4, the Value at Risk is thus

$$\mathrm{VaR}(c) = -Q^{\mathrm{cpf}_{\delta V}}_{1-c} \approx -\mu - \sqrt{\mu_2}\, Q^{\mathrm{cpf}_{\widetilde{\delta V}}}_{1-c}$$

  - for $Q^{\mathrm{cpf}_{\widetilde{\delta V}}}$ approximation 7.30 is used with the $(1-c)$ percentile of the standard normal distribution.

7.3.2 Fourier-Transformation of the Portfolio-Value Distribution

- Up to now the distribution itself has not been calculated directly.
- For this, characteristic functions (CFs) are neccessary.
- As in 6.11, the *characteristic function,* Φ_x, of a random variable x with density function $\mathrm{pdf}(x)$ is defined as the *Fourier transformation* of the density function[2]:

$$\Phi_x(s) \equiv \mathrm{E}[e^{isx}] = \int_{-\infty}^{\infty} e^{isx}\, \mathrm{pdf}(x) dx \tag{7.31}$$

[2] Here, i denotes the *imaginary number* satisfying the propertiy $i^2 = -1$, thus intuitively $i = \sqrt{-1}$.

– The CF of the $\mathrm{N}(0, \sum_{i \notin J} L_i^2)$-distributed random variable u_0 is:

$$\Phi_{u_0}(s) = \exp\left(-\frac{1}{2}s^2 \sum_{i \notin J} L_i^2\right) \tag{7.32}$$

– The CF of the $\chi^2(1, (L_j/\lambda_j)^2)$-distributed random variable u_j is:

$$\Phi_{u_j}(s) = \frac{1}{\sqrt{1-2is}} \exp\left\{\frac{is}{1-2is}\frac{L_j^2}{\lambda_j^2}\right\} \ , \ \ j \in J \ , \ \ i \equiv \sqrt{-1} \tag{7.33}$$

- CFs have similarly properties as MGFs:

$$\Phi_{x+y}(s) = \Phi_x(s)\,\Phi_y(s) \ , \ \ \Phi_{ax+b}(s) = e^{ibs}\Phi_x(as) \tag{7.34}$$

  for all non-stochastic values a, b and random variables x, y.

- Thus the CF of δV is

$$\Phi_{\delta V}(s) = \exp\left\{-is\sum_{j \in J}\frac{L_j^2}{2\lambda_j}\right\}\Phi_{u_0}(s)\prod_{j \in J}\Phi_{u_j}(\frac{1}{2}\lambda_j s) \tag{7.35}$$

- The result is of course the same as Equation 7.24 for the MGF with the obvious substitution $s \rightarrow is$:

$$\Phi_{\delta V}(s) = \exp\left(-\frac{1}{2}\, s^2 \sum_{i \notin J} L_i^2\right) \prod_{j \in J} \frac{1}{\sqrt{1 - i\lambda_j s}} \exp\left\{-\frac{1}{2} L_j^2 \frac{s^2}{1 - i\lambda_j s}\right\} \tag{7.36}$$

- Since $\lambda_i = 0$ for $i \notin J$,we can – as we did with the MGF – write the first exp-function as

$$\exp\left(-\frac{1}{2}\, s^2 \sum_{i \notin J} L_i^2\right) = \prod_{i \notin J} \frac{1}{\sqrt{1 - i\lambda_i s}} \exp\left\{-\frac{1}{2} L_i^2 \frac{s^2}{1 - i\lambda_i s}\right\}$$

  – Thus, the characteristic function can be written as a product over *all* indexes j:

$$\Phi_{\delta V}(s) = \prod_{j=1}^{n} \frac{1}{\sqrt{1 - i\lambda_j s}} \exp\left\{-\frac{1}{2} L_j^2 \frac{s^2}{1 - i\lambda_j s}\right\} \quad \text{with} \quad i \equiv \sqrt{-1} \tag{7.37}$$

- There exists an inverse transformation for the characteristic function, namely the *inverse Fourier Transformation*, see 6.12.

– Thus the density function $\mathrm{pdf}(\delta V)$ can be computed explicitly (at least numerically):

$$\begin{aligned}\mathrm{pdf}_{\delta V}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\Phi_{\delta V}(s)ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\prod_{j=1}^{n}\frac{1}{\sqrt{1-i\lambda_j s}}\exp\left\{-\frac{1}{2}L_j^2\frac{s^2}{1-i\lambda_j s}\right\}ds \qquad (7.38)\end{aligned}$$

– The *cumulative* distribution function of δV is obtained through the (numerical) integration of this probability density

$$\begin{aligned}\mathrm{cpf}_{\delta V}(c) &\equiv \int_{-\infty}^{c}\mathrm{pdf}_{\delta V}(x)dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{c}\int_{-\infty}^{\infty} e^{-isx}\prod_{j=1}^{n}\frac{\exp\left\{-\frac{1}{2}L_j^2\frac{s^2}{1-i\lambda_j s}\right\}}{\sqrt{1-i\lambda_j s}}ds\,dx\end{aligned}$$

- The recommended method for numerically performing the Fourier transformation and inverse Fourier transformation is the *fast Fourier transformation* (FFT).

- The FFT reduces the number of multiplications from order $O(N^2)$ to $O(N \ \ln(N))$.
- See, for example [4] oder [44].

7.3.3 Monte Carlo Simulations of the Portfolio-value Distribution

- All methods above offer sufficient possibilities for error.
  - Calculating the cumulative distribution of δV with characteristic functions involves complicated numerical procedures.
  - Using moment generating functions one needs additional assumptions and approximations to establish a relation between the moments and the distribution or the percentiles.
  - The Delta-Gamma method itself is only the second order Taylor approximation of the portfolio's value.
  - Significant difficulties and assumptions and approximations are often involved in calculating the Gamma and Covariance matrices.

- Hence a simple Monte Carlo simulation may not even be less accurate if a sufficient number of simulations are run.

  – Draw n standard normally distributed iid random numbers and compute the simulated change in the portfolio's value immediately from Equation 7.15:

  $$\delta V = \sum_{i=1}^{n} \left[L_i \, \delta X_i + \frac{1}{2} \lambda_i \delta X_i^2 \right]$$

  – Repat this procedure N times (several thousand times) to obtain N simulated changes δV.

  – The percentiles of the distribution can be approximated by simply sorting the simulated values of δV in increasing order.

  – A (complicated) full valuation of the portfolio is not necessary since a 2nd order proxy for the simulated value change is generated directly with Equation 7.15.

  – However, before the simulation can be performed,

    * the eigenvalues of the transformed Gamma matrix 7.4

* and the transformed sensitivities L_i first need to be determined.
* Because of Equations 7.17 and 7.16,
  · the Cholesky decomposition of the Covariance matrix as well as
  · the eigenvectors of the Gamma matrix must be computed.

8

Assigments

1. Show the name-giving property 6.4 of the Moment Generating Function, i.e. show that
$$\left.\frac{\partial^n \mathrm{E}[e^{sx}]}{\partial s^n}\right|_{s=0} = \mathrm{E}[x^n]$$
for any random variable x with distribution density $\mathrm{pdf}(x)$.

2. Show that the moments of the χ^2-distribution with n degrees of freedom are

$$\mathrm{E}[x^k]_{\chi^2(n)} = \prod_{i=0}^{k-1}(n+2i) \tag{8.1}$$

as stated in Equation 8.1.

3. Show that the transformed Gamma matrix $\mathbf{M}$ defined in Equation 7.2 is symmetric.

4. Show that the eigenvector matrix $\mathbf{O}$ defined in 7.7 is an orthonormal transformation, i.e. show that $\mathbf{O}^T\mathbf{O} = \mathbf{1}$.

5. Show that the covariances remain invariant under the transformation $\mathbf{O}^T$ and therefore the new random variables δX_i are independent:

$$\delta\mathbf{X} \sim \mathrm{N}(\mathbf{0}, \mathbf{1}) \ , \ \ iid$$

6. Caculate the portfolio variance in the Delta-Gamma approximation by explicitly differentiating Equation 7.27.

7. Caculate the third and fourth central moment in the Delta-Gamma approximation, thus proving Equation 7.29.

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