



# Martingales, Numeraires and Term Structure Models

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# Contents

<b>1</b>	<b>Martingale and Numeraire</b>	<b>1</b>
1.1	The Martingale Property . . . . .	1
1.2	The Numeraire . . . . .	5
1.3	Self-financing Portfolio Strategies . . . . .	13
<b>2</b>	<b>Deeper Insights</b>	<b>23</b>
2.1	The Generalization to Continuous Time . . . . .	23
2.2	The Drift . . . . .	42
2.3	The Market Price of Risk . . . . .	49

2.4	Tradable Underlyings . . . . .	53
2.5	Applications in the Black-Scholes World . . . . .	55
<b>3</b>	<b>Interest Rates and Term Structure Models</b>	<b>62</b>
3.1	Instantaneous Spot Rates and Instantaneous Forward Rates . . . . .	65
3.2	Important Numeraire Instruments . . . . .	69
3.2.1	The Risk-neutral Measure . . . . .	70
3.2.2	The Forward-neutral Measure . . . . .	74
3.3	Deterministic Interest Rates . . . . .	77
3.4	Tradeable and Non-Tradeable Instruments . . . . .	82
3.5	Convexity Adjustments . . . . .	85
3.5.1	LIBOR-In-Arrears Swaps . . . . .	92
3.5.2	Money Market Futures . . . . .	96
3.6	Arbitrage-free Interest Rate Trees . . . . .	101
3.6.1	Backward Induction . . . . .	103
3.6.2	Forward Induction and Greens-Functions . . . . .	113
3.7	Market Rates versus Instantaneous Rates . . . . .	125
3.7.1	Arrow-Debreu Prices . . . . .	127
3.7.2	Pricing Caplets by means of the Arrow-Debreu Prices . . . . .	134

3.8	Explicit Specification of Short Rate Models . . . . .	138
3.8.1	The Effect of the Volatility . . . . .	141
3.8.2	Normal Models . . . . .	143
3.8.3	Lognormal Models . . . . .	150
3.9	Explicit Step-by-Step Construction . . . . .	157
3.9.1	Absolute and Relative Volatilities . . . . .	163
3.9.2	Calibration of the Volatilities . . . . .	166
3.10	Monte Carlo on the Tree . . . . .	170
3.11	The Drift in Term Structure Models . . . . .	172
3.11.1	Heath-Jarrow-Morton Models . . . . .	173
3.11.2	Short Rate Models . . . . .	176
<b>4</b>	<b>Assigments</b>	<b>186</b>

# 1

## Martingale and Numeraire

### 1.1 The Martingale Property

- Ever heard these words of wisdom?

Today's price of a (tradeable) financial instrument is equal to the discounted expectation of its future price if the future expectation is

calculated with respect to the risk-neutral probability measure.

$$V_S(t) = B(t, T)E[V_S(T)] \quad (1.1)$$

- This holds for any financial instruments on any underlying  $S$  following a stochastic process of the general form

$$dS(t) = a(S, t) dt + b(S, t) dW \quad (1.2)$$

– where  $dW$  is a Brownian motion:

$$dW \sim X\sqrt{dt} \quad \text{with} \quad X \sim N(0, 1)$$

- We will now broadly expand on this concept ....
  - and show how right (and how wrong!) this is.
- *Risk-neutral probability* is an example of a so-called *martingale measure*.
  - Martingale measures are probability measures satisfying certain properties.

- but first: an intuitive explanation of the term "measure":
  - The expectation of a function  $f(X)$  of a random variable  $X$ , having a distribution with density function  $p$ , can be interpreted as an integral with respect to a certain *integral measure*:

$$\mathbb{E}[f(X)] = \int f(x) \underbrace{p(x)dx}_{\text{Integral measure}}$$

- Intuitively: By replacing the *Riemann integral measure*  $dx$  with the *probability measure*  $p(x)dx$ , one directly obtains the expectation of  $f(X)$  by simply integrating with respect to this *measure*.
- Consider now again Equation 1.1.

- Using  $B(T, T) = 1$  this can be written as

$$\frac{V_S(t)}{B(t, T)} = \mathbb{E} \left[ \frac{V(T)}{B(T, T)} \right]$$

– Defining the *normalized* option price by

$$Z_S(t) := \frac{V_s(t)}{B(t, T)}$$

– then

$$Z_S(t) = \mathbb{E}[Z_S(T)] \tag{1.3}$$

**The normalized price at time  $t$  is given by the expectation of the future normalized price<sup>1</sup>.**

- This is the *martingale property*:

**The *normalized* price is a martingale<sup>2</sup>.**

- *Normalizing* the price means

– expressing the price in units of zero-coupon bonds rather than in monetary units (like euros)

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<sup>1</sup>The expectation has to be taken with respect to the risk neutral measure.

<sup>2</sup>with respect to the risk neutral measure.

## 1.2 The Numeraire

- The normalizing instrument is commonly referred to as *numeraire*.
  - We now show: Not only zero bonds but *arbitrary* (tradable) instruments may serve as numeraires.
    - The numeraire used may not even refer to the underlying  $S$  (the zero bond does not do so either).
  - Definitions:
    - $S$ : the price of an underlying
    - $V_S$ : the price of an arbitrary financial instrument on this underlying
    - $Y$ : the price of a further, arbitrary, financial instrument to be used as a numeraire
      - \* above  $Y$  was chosen to be a zero bond
- Evolution of the underlying be (for now) described by a *binomial tree*.

- Construct a portfolio of  $\alpha$  underlyings and  $\beta$  financial instruments  $Y$

$$\Pi(t) = \alpha(t)S(t) + \beta(t)Y(t) \quad (1.4)$$

- Both the normalizing factor  $Y$  as well as the underlying  $S$  must be *tradable* to do this.H
  - however often the underlying *itself* is *not* tradable (for instance interest rates, see below!).
  - Then – instead of  $S$  – choose a tradable financial instrument,  $U_S$ , having  $S$  as its underlying.
  - The only restriction: it must not be possible to replicate  $U_S$  by the numeraire instrument  $Y$  (one needs two truly "linearly independent" instruments).
- Thus, our portfolio now looks slightly more general:

$$\Pi(t) = \alpha(t)U_S(t) + \beta(t)Y(t) \quad (1.5)$$

- Require that one time step later this portfolio shall have a value equal to that of the derivative  $V$  in all (two) states of the world:

$$\begin{aligned}\alpha(t)U_{S_u}(t + \delta t) + \beta(t)Y_u(t + \delta t) &= \Pi_u(t + \delta t) \stackrel{!}{=} V_{S_u}(t + \delta t) \\ \alpha(t)U_{S_d}(t + \delta t) + \beta(t)Y_d(t + \delta t) &= \Pi_d(t + \delta t) \stackrel{!}{=} V_{S_d}(t + \delta t)\end{aligned}\quad (1.6)$$

– The weights  $\alpha$  and  $\beta$  follow uniquely<sup>3</sup>.

$$\alpha(t) = \frac{V_{S_u}Y_d - V_{S_d}Y_u}{U_{S_u}Y_d - U_{S_d}Y_u}, \quad \beta(t) = \frac{V_{S_d}U_{S_u} - V_{S_u}U_{S_d}}{U_{S_u}Y_d - U_{S_d}Y_u}\quad (1.7)$$

- To prevent arbitrage portfolio at time  $t$  must also be equal to  $V$ :

$$\begin{aligned}V(t) = \Pi(t) &= \alpha(t)U_S(t) + \beta(t)Y(t) \\ &= \frac{V_{S_u}Y_d - V_{S_d}Y_u}{U_{S_u}Y_d - U_{S_d}Y_u}U_S(t) + \frac{V_{S_d}U_{S_u} - V_{S_u}U_{S_d}}{U_{S_u}Y_d - U_{S_d}Y_u}Y(t)\end{aligned}\quad (1.8)$$

---

<sup>3</sup>In order to simplify the notation, the time dependence is suppressed in the arguments. All variables equipped with an index  $u$  or  $d$  are evaluated at time  $t + \delta t$ .

– **Exactly at this point entered the assumption of an arbitrage-free market.**

- Collecting the coefficients of  $V_{S_u}$  and  $V_{S_d}$ :

$$V(t) = V_{S_u} \frac{Y_d U_S(t) - U_{S_d} Y(t)}{U_{S_u} Y_d - U_{S_d} Y_u} + V_{S_d} \frac{U_{S_u} Y(t) - Y_u U_S(t)}{U_{S_u} Y_d - U_{S_d} Y_u}$$

– to get the derivative value  $V$  normalized with the numeraire  $Y$  we rewrite this as

$$\frac{V(t)}{Y(t)} = \frac{V_{S_u}}{Y_u} \frac{Y_d U_S(t)/Y(t) - U_{S_d}}{U_{S_u} Y_d/Y_u - U_{S_d}} + \frac{V_{S_d}}{Y_d} \frac{U_{S_u} - Y_u U_S(t)/Y(t)}{U_{S_u} - U_{S_d} Y_u/Y_d}$$

- This can be written as

$$Z_S(t) = Z_S^u(t + \delta t)p_u + Z_S^d(t + \delta t)p_d \quad (1.9)$$

– by defining the normalized prices as

$$Z_S(t) := \frac{V(t)}{Y(t)} \quad , \quad Z_S^{u,d}(t + \delta t) := \frac{V_{S_{u,d}}(t + \delta t)}{Y_{u,d}(t + \delta t)}$$

– and the "probabilities" as<sup>4</sup>

$$\begin{aligned}
 p_u &:= \frac{Y_d \frac{U_S}{Y} - U_{S_d}}{U_{S_u} \frac{Y_d}{Y_u} - U_{S_d}} = \frac{Y_u Y_d U_S - Y_u Y U_{S_d}}{Y_d Y U_{S_u} - Y_u Y U_{S_d}} = \frac{\frac{U_S}{Y} - \frac{U_{S_d}}{Y_d}}{\frac{U_{S_u}}{Y_u} - \frac{U_{S_d}}{Y_d}} \\
 p_d &:= \frac{U_{S_u} - Y_u \frac{U_S}{Y}}{U_{S_u} - U_{S_d} \frac{Y_u}{Y_d}} = \frac{Y_d Y U_{S_u} - Y_d Y_u U_S}{Y_d Y U_{S_u} - Y_u Y U_{S_d}} = \frac{\frac{U_{S_u}}{Y_u} - \frac{U_S}{Y}}{\frac{U_{S_u}}{Y_u} - \frac{U_{S_d}}{Y_d}}
 \end{aligned} \tag{1.10}$$

\* They depend explicitly only on the numeraire  $Y$  and the instrument  $U_s$ .

\* Obviously (simple algebra):  $p_d = 1 - p_u$

- From 1.9 one can also write  $p$  as a function of  $Z_S$  (and thus of  $V_S$  and  $Y$ ):

$$p_u = \frac{Z_S(t) - Z_S^d(t + \delta t)}{Z_S^u(t + \delta t) - Z_S^d(t + \delta t)} \tag{1.11}$$

– In this form the independence of  $p$  on  $V_S$  is not immediately recognizable.

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<sup>4</sup>The last expressions are obtained by dividing both the numerator and denominator by  $Y_u Y_d Y$ .

- If  $p_u$  and  $p_d$  can be interpreted as probabilities, Equation 1.9 would be of the martingale form 1.3.
- The following conditions, holding for all probabilities, must be checked <sup>5</sup>:

$$p_u + p_d = 1 \quad , \quad p_u > 0 \quad , \quad p_d > 0 \quad (1.12)$$

- $p_u + p_d = 1$  follows immediately from simple algebra.
- $p_u$  and  $p_d$  have a *common* denominator  $\frac{U_{S_u}}{Y_u} - \frac{U_{S_d}}{Y_d}$ .
- This is  $> 0$  iff the normalized price of  $U_S$  in the "down state" is smaller than in the "up state".
  - \* Then: *both* numerators must be  $> 0$ , too.
- Denominator  $> 0$  not necessarily the case<sup>6</sup>, since "up" and "down" defined

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<sup>5</sup>If all three of these conditions hold, it follows immediately that  $p_u < 1$  and  $p_d < 1$  as well.

<sup>6</sup>Even when  $S$  is tradable, allowing  $U_S$  to be replaced by the underlying, there are several common instruments that violate this condition when used as a normalizing instrument. For example, the value  $C$  of a plain vanilla call on  $S$  increases *faster* (in percentage terms) than  $S$  itself implying for the quotient  $\frac{S}{C(S)}$  that  $\frac{S_2}{C(S_2)} < \frac{S_1}{C(S_1)}$  for  $S_2 > S_1$ .

through the *unnormalized underlying* price  $S$  ( $S_u > S_d$  by definition) and not through the normalized price  $U_S/Y$ .

\* if denominator  $< 0$  then *both* numerators must be  $< 0$ , too.

– In summary:

$$\begin{aligned} \frac{U_{S_u}}{Y_u} > \frac{U_S}{Y} > \frac{U_{S_d}}{Y_d} \quad \text{for } S_u > S > S_d \\ \text{or} \\ \frac{U_{S_u}}{Y_u} < \frac{U_S}{Y} < \frac{U_{S_d}}{Y_d} \quad \text{for } S_u > S > S_d \end{aligned} \tag{1.13}$$

• The normalized price  $U_S$  must be a strictly monotone function of the underlying.

– If not: arbitrage opportunity!

– Assume for instance  $\frac{U_{S_u}}{Y_u} < \frac{U_S}{Y} > \frac{U_{S_d}}{Y_d}$ .

\* Exploit this by

· selling (short selling) the instrument  $U_S$  at time  $t$  and

- using the proceeds to purchase  $a = U_S/Y$  of the instrument  $Y$ .
- \* For this (again) both  $Y$  and  $U_S$  must be *tradable*!
- \* This portfolio has a value at time  $t$  of

$$-U_S + aY = -U_S + \left(\frac{U_S}{Y}\right)Y = 0$$

- \* One time step later, the portfolio's value is, in all events  $u$  and  $d$ , positive:

$$-U_{S_u} + aY_u = -U_{S_u} + \left(\frac{U_S}{Y}\right)Y_u = Y_u \underbrace{\left[-\frac{U_{S_u}}{Y_u} + \frac{U_S}{Y}\right]}_{>0} > 0$$

$$-U_{S_d} + aY_d = -U_{S_d} + \left(\frac{U_S}{Y}\right)Y_d = Y_d \underbrace{\left[-\frac{U_{S_d}}{Y_d} + \frac{U_S}{Y}\right]}_{>0} > 0$$

- \* That's a *certain* profit without any capital or risk.
- Conversely, if  $\frac{U_{S_u}}{Y_u} > \frac{U_S}{Y} < \frac{U_{S_d}}{Y_d}$ , analogous arbitrage opportunity by going long in  $U_S$  and short in  $a = U_S/Y$  of the instrument  $Y$ .

- If the market is arbitrage-free, the conditions 1.13 are *automatically* satisfied for all tradable financial instruments.
  - Therefore these conditions need not be verified in practice.
  - In arbitrage free markets  $p_u$  and  $p_d$  in 1.10 are in deed probabilities.
  - Thus, the normalized price  $Z = V_S/Y$  (and thus the normalized prices of all tradable instruments on  $S$ ) is a martingale.
- Or conversely: if there is a tradable instrument  $U_S$  on  $S$  whose normalized price is not a martingale, a portfolio consisting of  $U_S$  and  $Y$  could be constructed making arbitrage possible.

### 1.3 Self-financing Portfolio Strategies

- Up to now: one single, discrete time-step.
- The extension to arbitrarily many discrete time steps is completely analogous.
- At every node of the tree construct a portfolio

- with the numeraire  $Y$  and
  - a financial instrument  $U_S$  on the underlying (or the underlying itself if it is tradable)
  - to replicate both possible derivative prices  $V$  in the next step.
- Such a replication is a *hedge* of  $V$ .
  - From one time step to the next, this hedge is constantly adjusted by adjusting the weights  $\alpha(t)$  and  $\beta(t)$  of  $U_S$  and  $Y$ .
  - The weights at a specific time  $t$  are determined from  $V, Y$  and  $U_S$  one time step *later*; see 1.7.
  - One thus goes backwards through the tree from some future time point  $T$  at which the price  $V(T)$  is known.
  - Then, the price of the replicating portfolio at the first node is the desired price of the derivative
    - **if** the position adjustments made from step to step were accomplished *without* the infusion or removal of capital.

- To guarantee this, consider the replicating portfolio at time  $t$

$$\Pi(t) = \alpha(t)U_S(t) + \beta(t)Y(t)$$

- This is *set up* so that it equals  $V$  at this time point (see 1.8):

$$\Pi(t) = V(t)$$

- Likewise, the portfolio set up at time  $t + \delta t$

$$\Pi(t + \delta t) = \alpha(t + \delta t)U_S(t + \delta t) + \beta(t + \delta t)Y(t + \delta t)$$

- is *set up* so that it equals  $V$  at *this* time point:

$$\Pi(t + \delta t) = V(t + \delta t)$$

- Now consider the value of the portfolio at time  $t + \delta t$  which was set up at time  $t$ .

- denote this value by  $\Pi_t(t + \delta t)$ : the index  $t$  indicates that this value refers to the portfolio *set up* at time  $t$ .

- From the construction Equation 1.6, this has, in all events, the same value as  $V$  at time  $t + \delta t$ .
- The *new* portfolio  $\Pi_{t+\delta t}$ , set up at time  $t + \delta t$  also has the value  $V(t + \delta t)$ , as required by 1.8.
- Thus, at time  $t + \delta t$  the value  $\Pi_t$  of the *old* portfolio is by construction exactly equal to the value  $\Pi_{t+\delta t}$  of the *new* portfolio.
  - one could dissolve the old portfolio  $\Pi_t$  at time  $t + \delta t$  and use the proceeds to construct the new portfolio  $\Pi_{t+\delta t}$ .
  - No capital flows either into or out of the portfolio.
  - Thus, the price at the first node required to construct the replicating portfolio is in fact the the price of the derivative  $V$
- Such a strategy is called *self financing*.
- Reconsider this from another point of view.

- The total difference  $\delta\Pi(t)$  in the portfolio value for *any* trading strategy over a time span  $\delta t$  equals the difference

- in the value at time  $t + \delta t$  of the portfolio set up a time  $t + \delta t$
- and the value at time  $t$  of the portfolio set up at time  $t$

$$\begin{aligned}\delta\Pi(t) &= \Pi_{t+\delta t}(t + \delta t) - \Pi_t(t) \\ &= \underbrace{\Pi_t(t + \delta t) - \Pi_t(t)}_{\delta\Pi^{\text{Market}}} + \underbrace{\Pi_{t+\delta t}(t + \delta t) - \Pi_t(t + \delta t)}_{\delta\Pi^{\text{Trading}}}\end{aligned}$$

- \* One arrives at the second equation by simply inserting  $0 = \Pi_t(t + \delta t) - \Pi_t(t + \delta t)$ .

- two contributions to the total change in  $\Pi$ :

- $\delta\Pi^{\text{Market}}$  is the change resulting from changes in the market without adjusting the position.
- $\delta\Pi^{\text{Trading}}$  is the difference a time  $t + \delta t$  between the value of the new portfolio and that of the old portfolio, i.e. the value change resulting solely from trading.

- For a *self-financing* strategy  $\delta\Pi^{\text{Trading}}$  must be zero since the old portfolio must provide exactly the funds necessary to finance the new portfolio.
- Example: the portfolio  $\Pi(t) = \alpha(t)U_S(t) + \beta(t)Y(t)$  as above:

$$\begin{aligned}
\delta\Pi^{\text{Market}} &= \Pi_t(t + \delta t) - \Pi_t(t) \\
&= \alpha(t)\underbrace{[U_S(t + \delta t) - U_S(t)]}_{\delta U_S(t)} + \beta(t)\underbrace{[Y(t + \delta t) - Y(t)]}_{\delta Y(t)} \\
\delta\Pi^{\text{Trading}} &= \Pi_{t+\delta t}(t + \delta t) - \Pi_t(t + \delta t) \\
&= U_S(t + \delta t)\underbrace{[\alpha(t + \delta t) - \alpha(t)]}_{\delta\alpha(t)} + Y(t + \delta t)\underbrace{[\beta(t + \delta t) - \beta(t)]}_{\delta\beta(t)}
\end{aligned}$$

– Then

$$\begin{aligned}
\delta\Pi(t) &= \alpha(t)\delta U_S(t) + \beta(t)\delta Y(t) + U_S(t + \delta t)\delta\alpha(t) + Y(t + \delta t)\delta\beta(t) \\
&= \alpha(t)\delta U_S(t) + \beta(t)\delta Y(t) \quad \text{for a self-financing strategy} \quad (1.14)
\end{aligned}$$

**A strategy is self-financing if and only if the *total* change in the portfolio's value over time can be explained exclusively by market changes.**

- This holds for infinitesimal time steps as well:

$$\begin{aligned}\Pi(t) &= \alpha(t)U_S(t) + \beta(t)Y(t) && \text{self-financing strategy} \\ &\iff \\ d\Pi(t) &= \alpha(t)dU_S(t) + \beta(t)dY(t) && (1.15)\end{aligned}$$

- The weights  $\alpha(t)$  and  $\beta(t)$ , necessary to replicate the derivative one time step *later* are already known at time  $t$ .
  - A process whose value at a specific time is already known at the previous time step is called a *previsible process*.
- Structure of Equation 1.14:
  - The (at time  $t$  unknown) change  $\delta\Pi$  over the next time step is composed of
    - \* the (at time  $t$  unknown) change  $\delta U$  and

- \* the (at time  $t$  unknown) change  $\delta Y$ .
- \* the (at time  $t$  **known**) *coefficients*  $\alpha(t)$  and  $\beta(t)$  of these changes.

- Remark: All stochastic models (except for jump models) you encounter are Ito processes of the general form

$$dS(t) = a(S, t) dt + b(S, t) dW \quad \text{with} \quad dW \sim X\sqrt{dt} \quad , \quad X \sim N(0, 1) \quad (1.16)$$

- The coefficients  $a(S, t)$  and  $b(S, t)$  which control the next step in the process are also already known at time  $t$ .
  - These processes  $a$  and  $b$  are also *previsible*.
- Summary of the deep and important concepts presented above:
    - For any arbitrary underlying  $S$  which follows a stochastic process of the general form indicated in Equation 1.16,
    - and any tradable instrument  $U$  with  $S$  as its underlying (or  $S$  itself if it is tradable)
    - and any other arbitrary, tradable, financial instrument  $Y$  (*numeraire*),

- the assumption of an arbitrage-free market implies the existence of a (numeraire-dependent) unique probability *measure*  $p$ ,
  - such that the current, normalized price  $Z_S = V_S/Y$  (also called the *relative price*) of an arbitrary, tradable, financial instrument  $V_S$  on  $S$
  - equals the expectation of the future normalized price and thus,  $Z_S$  is a martingale with respect to  $p$ .
  - This statement holds because a *self-financing* portfolio strategy can be followed with *previsible* weights which replicates (*hedges*) the price of the financial instrument  $V_S$  at all times.
- The expectation is taken at time  $t$  with respect to the  $Y$ -dependent probability measure  $p$  given explicitly by 1.10 (in the context of a binomial tree over one time step)
  - To emphasize this dependence on  $t$  and  $Y$  we write:

$$Z_S(t) = \mathbb{E}_t^Y [Z_S(u)] \quad \forall u \geq t \quad (1.17)$$

- Intuitive interpretation:

**In the martingale measure the expected change in the normalized price of a tradable instrument is zero, i.e, the normalized price has no drift.**

- I.e., in this measure a normalized price process is of the form 1.16 with no drift term:

$$dZ_S(t) = \tilde{g}_Z(S, t) d\tilde{W} \quad \text{with} \quad d\tilde{W} \sim X\sqrt{dt} \quad , \quad X \sim N(0, 1) \quad (1.18)$$

- Here,  $d\tilde{W}$  is Brownian motion and  $\tilde{g}_Z$  a (*previsible*) process which is different for each instrument.

# 2

## Deeper Insights

### 2.1 The Generalization to Continuous Time

- The above statements hold for the time-continuous case as well.
- But several fundamental theorems from stochastic analysis are required.

- In the following the necessary theorems from stochastic analysis will be presented (without proof) and used when needed.
  - This will provide a much deeper understanding, in particular of role of the *drift* of an underlying.
- Consider a general (not necessarily tradable) underlying  $S$ , which, *in the real world*, is governed by an Ito process satisfying Equation 1.16, i.e.,

$$dS(t) = a(S, t) dt + b(S, t) dW \quad \text{with} \quad dW \sim X\sqrt{dt} \quad , \quad X \sim N(0, 1) \quad (2.1)$$

- Let  $U(S, t)$  denote the price of a tradable instrument with and underlying  $S$ .
- The process for  $U$  *in the real world* is (from Ito's Lemma):

$$dU(S, t) = a_U(S, t)dt + \frac{\partial U}{\partial S} b(S, t) dW \quad \text{with}$$

$$a_U(S, t) := \frac{\partial U}{\partial S} a(S, t) + \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} b(S, t)^2 \quad (2.2)$$

- Select an arbitrary, tradable instrument  $Y$  as the numeraire.

- The numeraire  $Y$  is however not completely arbitrary.
  - We always have assumed that our market is driven by only one *single* random factor (*one-factor model*), the *Brownian motion*  $dW$  in 1.16.
  - Thus if the numeraire has a stochastic component, it must be driven by *the same* random walk as the underlying  $S$ .

- Therefore the most general process describing for the numeraire is,

$$dY(t) = m(Y, t) dt + n(Y, t) dW \quad \text{with} \quad dW = X\sqrt{dt}, X \sim N(0, 1) \quad (2.3)$$

- with *previsible* processes  $m$  and  $n$  and the same random walk  $dW$ , as in 2.1.
- We seek a probability measure with respect to which the prices of all tradable instruments normed with the numeraire  $Y$  are martingales.
- We start by trying to find a measure for which the normed price

$$Z(S, t) := \frac{U(S, t)}{Y(t)} \quad (2.4)$$

of the *single* selected instrument  $U$  is a martingale.

- The product rule yields (Assignment!)<sup>1</sup>:

$$dZ = d[Y^{-1}U] = Ud[Y^{-1}] + Y^{-1}dU + dUd[Y^{-1}] \quad (2.5)$$

- The differential  $dU$  was already specified above.
- The differential  $d[Y^{-1}]$  is explicitly (Assignment!)

$$d[Y^{-1}] = \left[ -\frac{1}{Y^2}m + \frac{1}{Y^3}n^2 \right] dt - \frac{1}{Y^2}n dW$$

- Collecting terms only up to order *linear* in  $dt$  yields reduces the last term in 2.5 to:

$$\begin{aligned} dUd[Y^{-1}] &= \left( a_U dt + \frac{\partial U}{\partial S} b dW \right) \frac{1}{Y^2} ([-m + n^2/Y] dt - n dW) \\ &= -\frac{\partial U}{\partial S} b \frac{n}{Y^2} \underbrace{(dW)^2}_{dt} + \mathcal{O}(dtdW) \end{aligned}$$

---

<sup>1</sup>We suppress the arguments of  $U, Y, a, b, m$  and  $n$  in order to keep the notation simple. The arguments of these variables are always those as given in Equations 2.1 and 2.3.

- Altogether  $dZ$  becomes

$$\begin{aligned}
dZ &= \frac{U}{Y} \left( \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] dt - \frac{n}{Y} dW \right) + \frac{a_U}{Y} dt + \frac{\partial U}{\partial S} \frac{b}{Y} dW - \frac{\partial U}{\partial S} \frac{bn}{Y^2} dt \\
&= \left( \frac{b}{Y} \frac{\partial U}{\partial S} - \frac{n}{Y} \frac{U}{Y} \right) dW + \left( \frac{a_U}{Y} + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] \frac{U}{Y} - \frac{bn}{Y^2} \frac{\partial U}{\partial S} \right) dt \\
&= \left( \frac{b}{Y} \frac{\partial U}{\partial S} - \frac{n}{Y} \frac{U}{Y} \right) \left\{ dW + \frac{a_u + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U - b \frac{n}{Y} \frac{\partial U}{\partial S}}{b \frac{\partial U}{\partial S} - n \frac{U}{Y}} dt \right\} \quad (2.6)
\end{aligned}$$

– The coefficient of  $dW$  was factored out "by force" in the final step.

- We seek a probability measure with respect to which  $Z$  is a martingale.
  - For this,  $dZ$  would have to have no drift, i.e. be of the form 1.18.
  - $dZ$  would have the desired form if there would exist a measure with respect to which

$$d\widetilde{W} := dW + \frac{a_u + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U - b \frac{n}{Y} \frac{\partial U}{\partial S}}{b \frac{\partial U}{\partial S} - n \frac{U}{Y}} dt \quad (2.7)$$

were a standard Brownian motion, i.e.  $d\widetilde{W} \sim X\sqrt{dt}$  with  $X \sim N(0,1)$  in this measure.

- Stochastic analysis delivers a theorem to guarantee the existence of such a measure:

**Theorem 1 (Girsanov)** *Let  $W(t)$  be a Brownian motion with respect to the probability measure  $\mathcal{P}$  and  $\gamma(t)$  a previsible process which (for some future time  $T$ ) satisfied the boundedness condition*

$$\mathbb{E}_{\mathcal{P}} \left[ \exp \left( \frac{1}{2} \int_0^T \gamma(t) dt \right) \right] < \infty$$

*then there exists a measure  $\mathcal{Q}$ , equivalent to  $\mathcal{P}$ <sup>2</sup>, with respect to which*

$$\widetilde{W}(t) = W(t) + \int_0^t \gamma(s) ds$$

---

<sup>2</sup>Two probability measures are called equivalent if they agree exactly on what is possible and what is impossible. I.e. an event is impossible (probability zero) w.r.t. one probability measure if and only if it is impossible w.r.t. all equivalent probability measures.

is a Brownian motion. This implies that

$$dW(t) + \gamma(t)dt = d\widetilde{W}(t) \sim X\sqrt{dt} \text{ with } X \sim N(0, 1)$$

Conversely, in the measure  $\mathcal{Q}$  the original process  $W(t)$  is a Brownian motion with an additional drift component,  $-\gamma(t) : dW(t) = d\widetilde{W}(t) - \gamma(t)dt$

- To apply the theorem, identify the coefficient of  $dt$  in 2.7 with  $\gamma(t)$ .
  - Observe that this coefficient of  $dt$  in 2.7 is previsible, since everything in it is known at time  $t$ .
  - We just *assume* (as always) in our models the technical boundedness condition in the theorem to be satisfied.
- Thus the theorem holds and there indeed exists a measure in which  $d\widetilde{W}$  is a simple Brownian motion.
  - In this measure  $dZ$  has the form in 1.18 with

$$\widetilde{g}_Z(S, t) = \frac{b}{Y} \frac{\partial U}{\partial S} - \frac{n}{Y} \frac{U}{Y} \tag{2.8}$$

- Can we now conclude that  $Z$  is a martingale with respect to this measure?
  - Does the absence of a drift directly imply<sup>3</sup> the martingale property<sup>1.3</sup>?
  - Again, stochastic analysis provides theorems which guarantee (if certain technical conditions are satisfied, see [1, Seite 79], for example) that this is the case.
  - The measure for which  $dZ$  has the form 1.18 is thus a martingale measure.
- We have found a measure with respect to which the normed price of *one* selected instrument  $U$  (or for  $S$  if  $S$  should be tradable) is a martingale.
  - The only requirement made on the instrument  $U$  is that it be tradable.
  - Thus, for every arbitrary, tradable financial instrument, (with  $S$  as an underlying) there exists a martingale measure.
- Is this measure different for each instrument  $U$ ?

---

<sup>3</sup>We have already only shown the reverse implication above.

- Does it depend on our choice of  $U$  (just as it depends on the choice of numeraire  $Y$ )?
- It thus remains to show that the normed price of *every* tradable instrument (with  $S$  as its underlying) is a martingale with respect to *the same* probability measure.
  - In the discrete case, the essential point was the self-financing strategy for a portfolio consisting of  $U$  and  $Y$  which replicates  $V$  exactly.
  - We will need another important theorem from stochastic analysis to establish this for the time-continuous case:

**Theorem 2 (Martingale Representation)** *If  $Z$  is a martingale with respect to the probability measure  $\mathcal{P}$  with a volatility which is non-zero almost surely (with probability  $P = 1$ ), i.e. if  $Z$  follows a stochastic process satisfying*

$$dZ = b_Z(t)dW \quad \text{with} \quad P[b_Z(t) \neq 0] = 1 \quad \forall t$$

*with a previsible process  $b_Z(t)$ , and if there exists in this measure another martingale  $X$ , then there exists a previsible process  $\alpha(t)$  such that*

$$dX = \alpha(t)dZ$$

Or equivalently in integral form

$$X(t) = X(0) + \int_0^t \alpha(s) dZ(s)$$

The process  $\alpha(t)$  is unique. Furthermore,  $\alpha$  and  $b_Z$  together satisfy the boundedness condition

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \alpha(t)^2 b_Z(t)^2 dt \right) \right] < \infty$$

- Intuitively: If the volatility is (almost surely) non-zero, two martingales differ at most by a previsible process.
- As yet, we only have one martingale namely  $Z$ , the normed price of  $U$ .
- We require a second martingale.
  - Since we wish to get information about the financial instrument  $V$ , construct a second martingale from  $V$ .
  - We do this with yet another quite simple theorem:

**Theorem 3 (Tower Law)** For any arbitrary function  $V$ , depending on events occurring up to some specified future time  $T > t$ , the expectation at time  $t$  of  $V(T)$  with respect to any arbitrary probability measure  $\mathcal{P}$ ,

$$E(t) := E_t^{\mathcal{P}} [V(T)]$$

is a martingale with respect to  $\mathcal{P}$ , explicitly

$$E(t) = E_t^{\mathcal{P}} [E(u)] \quad \forall u > t$$

- Proof is easy: Jus substitute the definition of  $E$  into the claim that  $E(t)$  is a martingale:

$$E_t^{\mathcal{P}} [V(T)] = E_t^{\mathcal{P}} [E_u^{\mathcal{P}} [V(T)]] \quad \forall u > t \quad (2.9)$$

– In words:

- \* taking the expectation at time  $u$  and subsequently taking the expectation of the result at an earlier time  $t$
- \* is the same as
- \* taking the expectation at the earlier time  $t$  from the start.

- The payoff profile  $V(T)$  of any financial instrument with maturity  $T$  is a function depending only on events (values of the underlying process  $S$ ) occurring up to the time  $T$ .
  - Thus, by the Tower Law, the *expectation* of this payoff profile is a martingale with respect to *every* probability measure.
- Thus, we have found two processes which are martingales in the same measure:  $Z(t)$  and  $\mathbb{E}_t^Y [V(T)]$ .
- We want more, however: We want the price  $V(t)$  *itself* to be a martingale not merely the *expectation* of  $V(T)$ .
- In the discrete case (and in the continuous case for  $U$  as well) the *normed* prices are martingales.
- Therefore consider the payoff normed with  $Y$ , ie.  $V(T)/Y(T)$ .
- The expectation of this is, because of the Tower Law, also a martingale:

$$E(t) := \mathbb{E}_t^Y \left[ \frac{V(T)}{Y(T)} \right]$$

- Observe that the payoff  $V(T)$  is exactly replicated by  $Y(t)E(t)$ :

$$Y(T)E(T) = Y(T)\mathbf{E}_T^Y \left[ \frac{V(T)}{Y(T)} \right] = Y(T)\frac{V(T)}{Y(T)} = V(T) \quad (2.10)$$

- It should now be clear how the existence of a replicating portfolio can be established through the martingale representation theorem:
- As one martingale we take  $E(t)$ , the expectation of the normed derivative price at maturity
- as the other martingale we take  $Z(t)$  from Equation 2.4, the normed price of the initially selected, tradable instrument  $U$ .
- The martingale representation theorem now states that (if the volatility of  $Z$  is always non-zero) the process  $E(t)$  differs from the process  $Z(t)$  only by a *previsible* process  $\alpha(t)$ :

$$dE = \alpha(t)dZ \quad (2.11)$$

- We use this previsible process to construct a portfolio consisting of  $\alpha(t)$  of the instrument  $U$  and  $\beta(t)$  of the numeraire  $Y$ , similar to Equation 1.5.

- This is always possible since  $\alpha(t)$  is previsible by the Martingale Representation Theorem and both  $U$  and  $Y$  are tradable.

$$\Pi(t) = \alpha(t)U(t) + \beta(t)Y(t) \quad (2.12)$$

- We require this portfolio to be equal  $Y(t)E(t)$  for all times  $t \leq T$ .

– Then, according to 2.10, it exactly replicates the payoff of  $V$  at maturity.

- From this condition, we can derive the number  $\beta(t)$  of numeraire instruments required:

$$\begin{aligned} Y(t)E(t) = \Pi(t) &= \alpha(t)U(t) + \beta(t)Y(t) \\ &\iff \\ \beta(t) &= E(t) - \alpha(t)\frac{U(t)}{Y(t)} = E(t) - \alpha(t)Z(t) \end{aligned} \quad (2.13)$$

- We have thus established the existence of a replicating portfolio.
- It remains to show that this portfolio is *self-financing*.

- Consider the total change in the portfolio value:

$$\begin{aligned}
 d\Pi &= d(YE) \\
 &= EdY + YdE + dEdY \\
 &= EdY + Y\alpha dZ + \alpha dZdY \\
 &= [\beta + \alpha Z] dY + \alpha Y dZ + \alpha dZdY \\
 &= \alpha [ZdY + YdZ + dZdY] + \beta dY \\
 &= \alpha \underbrace{d(ZY)}_U + \beta dY
 \end{aligned}$$

- in the second equality the product rule has been applied,
- in the third equality the Martingale Representation Theorem in form of Equation 2.11,
- and in the fourth, Equation 2.13 in the form  $E(t) = \beta(t) + \alpha(t)Z(t)$  has been used.
- In the last equation the product rule has been applied again.

- The last line now states that the total change in the portfolio results solely from the price changes of the instruments  $U$  and  $Y$  and *not* through any adjustment of the positions  $\alpha$  or  $\beta$ :

$$d\Pi(t) = \alpha(t)dU(t) + \beta(t)dY(t) \quad (2.14)$$

- Thus, via Equation 1.15 the portfolio is *self-financing*.
- The portfolio is by construction  $\Pi(t) = Y(t)E(t)$  for all times.
- At time  $T$  the portfolio, by 2.10, exactly replicates the payoff  $V(T)$  .
- The value of the portfolio must therefore equal  $V$  for all previous times as well:

$$V(t) = \Pi(t) = Y(t)E(t) = Y(t)\mathbf{E}_t^Y \left[ \frac{V(T)}{Y(T)} \right] \quad (2.15)$$

- and thus

$$\frac{V(t)}{Y(t)} = \mathbf{E}_t^Y \left[ \frac{V(T)}{Y(T)} \right] \quad (2.16)$$

- Therefore: The normed price of *all* tradable instruments are martingales in the *same* probability measure with respect to which the normed price of the instrument  $U$  is a martingale.
- The process  $\alpha(t)$  can be calculated explicitly.
  - Equation 2.11 states:  $\alpha(t)$  is the change of  $E$  per change in  $Z$  or: the derivative of  $E$  with respect to  $Z$ .
  - With 2.4 and 2.15,  $Z$  and  $E$  can be expressed in terms of the prices of tradable instruments (known at time  $t$ )

$$\alpha(t) = \frac{\partial E(t)}{\partial Z(t)} = \frac{\partial [V(t)/Y(t)]}{\partial [U(t)/Y(t)]} \quad (2.17)$$

- With Equation 2.13 one can now calculate  $\beta(t)$  explicitly as well:

$$\beta(t) = E(t) - \alpha(t)Z(t) = \frac{V(S,t)}{Y(t)} - \frac{U(S,t)}{Y(t)} \frac{\partial [V(S,t)/Y(t)]}{\partial [U(S,t)/Y(t)]}$$

- We have now shown that the normed prices  $V/Y$  of *all* tradable instruments are martingales with respect to the *same* martingale measure, namely the martingale measure of  $Z = U/Y$ .
- The only question remaining: Is this measure unique?
- Yet another theorem from stochastic analysis states that for so-called *complete markets*<sup>4</sup> the martingale measure is indeed unique[14]:

**Theorem 4 (Harrison-Pliska)** *A market consisting of financial instruments and a numeraire instrument is arbitrage-free if and only if there exists a measure with respect to which the prices of all financial instruments normed with the numeraire instrument are martingales. This measure is unique if and only if the market is complete.*

## Summary

The summary corresponds to the summary at the beginning of the Section 2.1 (which was done for discrete time steps).

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<sup>4</sup>A market is called *complete* if there exists a replicating portfolio for each financial instrument in the market.

- We select a tradable instrument  $Y$  as the numeraire and a further tradable instrument  $U$ , which has  $S$  as an underlying (if  $S$  itself is tradable,  $S$  can be chosen directly).
- We then find the probability measure for which  $Z = U/Y$  is a martingale. The *Girsanov-Theorem* guarantees that this is always possible via a suitable drift transformation as long as a technical boundedness condition is satisfied.
- The *Martingale Representation Theorem* and the *Tower Law* enable the construction of a *self-financing portfolio* composed of the instruments  $U$  and  $Y$  which replicates the payoff profile at maturity of any arbitrary, tradable instrument  $V$  having  $S$  as an underlying. The value of this replicating portfolio is given by Equation 2.15 where the expectation is taken with respect to the martingale measure of  $Z = U_S/Y$ .
- This portfolio must be equal to the value of the derivative  $V(t)$  for all previous times if the market is arbitrage-free. This means that, according to Equation 2.16, the normed price  $V/Y$  with respect to the martingale measure of  $Z$  is likewise a martingale. Thus, having obtained (via Girsanov) a martingale measure

for  $Z$ , the normed price  $V/Y$  of *all other* tradable instruments are martingales with respect to this measure.

- Finally, the *Theorem of Harrison & Pliska* states that this measure is unique in complete markets: In complete markets, there exists for every numeraire instrument, *one single* measure with respect to which all tradable instruments normed with this numeraire are martingales.

## 2.2 The Drift

- With respect to the martingale measure, the price processes of all instruments normed with the numeraire are *drift-free*.
- What can we say about the process of the *underlying* with respect to this measure?
- The model 1.16 describes the underlying in the *real* world. How does this look in the martingal measure?

- Equations 2.7 and 2.8 show the explicit relationships between the variables in the real world and the world governed by the martingale measure.
- Via 2.7 express the underlying process 2.1 in terms of the Brownian motion  $d\widetilde{W}$  (using  $a_u$  from 2.2):

$$\begin{aligned}
dS &= a dt + b dW \\
&= a dt + b \left\{ d\widetilde{W} - \frac{a_u + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U - b \frac{n}{Y} \frac{\partial U}{\partial S}}{b \frac{\partial U}{\partial S} - n \frac{U}{Y}} dt \right\} \\
&= a dt - \frac{\frac{\partial U}{\partial S} a + \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} b^2 + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U - b \frac{n}{Y} \frac{\partial U}{\partial S}}{\frac{\partial U}{\partial S} - \frac{n}{b} \frac{U}{Y}} dt + b d\widetilde{W} \\
&= \frac{b \frac{n}{Y} \frac{\partial U}{\partial S} - a \frac{n}{b} \frac{U}{Y} - \frac{\partial U}{\partial t} - \frac{1}{2} \frac{\partial^2 U}{\partial S^2} b^2 - \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U}{\frac{\partial U}{\partial S} - \frac{n}{b} \frac{U}{Y}} dt + b d\widetilde{W} \tag{2.18}
\end{aligned}$$

- This equation explicitly specifies the underlying process with respect to the martingale measure.

- Consistent with the Girsanov Theorem, in the transition from the real world measure to the martingale measure, only the drift of the underlying has changed and *not* the volatility.
  - \* The coefficient of the Brownian motion remains  $b(S, t)$ .
- From the Harrison-Pliska-Theorem: the martingale measure in complete markets (we will in future always assume that the market is complete) is *unique*. Thus the drift of  $S$ , used in the valuation of financial instruments is *unique* as well for every numeraire  $Y$ .
- In the last equality in 2.18 the terms  $\sim \frac{\partial U}{\partial S} a$  cancel and the real world drift enters only in the form of  $\frac{n}{b} \frac{U}{Y} a$ .
- The underlying drift  $a$  of the *real* world *disappears completely* for numeraires  $Y$  satisfying 2.3 with  $n(Y, t) = 0 \forall Y, t$ , i.e. for numeraires with processes of the form

$$dY(t) = m(Y, t) dt \tag{2.19}$$

- This by no means implies that  $Y$  must be deterministic, since  $m(Y, t)$  is not

assumed to be deterministic but only *previsible*<sup>5</sup>.

- The evolution of  $Y$  is at time  $t$  only known for the step *immediately following*  $t$ , not however for later steps.
- The numeraire should always be chosen to be of the form 2.19 for some previsible process  $m$ .

\* This is always possible in practice.

- For such numeraires,  $m/Y$  in Equation 2.18 is precisely the *yield of the numeraire instrument*, since Equation 2.19 obviously implies:

$$m = \frac{dY}{dt} \quad \Longrightarrow \quad \frac{m}{Y} = \frac{1}{Y} \frac{dY}{dt} = \frac{d \ln Y}{dt} \quad (2.20)$$

---

<sup>5</sup>A previsible process is a *stochastic* process whose current value can be determined from information available at the previous time step. Intuitively, it is a stochastic process "shifted" back one step in time.

- Thus, with  $n = 0$ , the drift transformation 2.7 becomes

$$\begin{aligned}
d\widetilde{W} &= dW + \frac{a_U - \frac{d \ln Y}{dt} U}{b \frac{\partial U}{\partial S}} dt \\
&= dW + \frac{1}{b} \left[ a_U \left( \frac{\partial U}{\partial S} \right)^{-1} - \frac{d \ln Y}{dt} \left( \frac{\partial \ln U}{\partial S} \right)^{-1} \right] dt \quad (2.21)
\end{aligned}$$

- The underlying process 2.18 reduces to

$$\begin{aligned}
dS(t) &= \widetilde{a}(S, t) dt + b(S, t) d\widetilde{W} \quad \text{with} \\
\widetilde{a}(S, t) &= \left( \frac{\partial U(S, t)}{\partial S} \right)^{-1} \left[ U(S, t) \frac{d \ln Y(t)}{dt} - \frac{\partial U(S, t)}{\partial t} - \frac{b(S, t)^2}{2} \frac{\partial^2 U(S, t)}{\partial S^2} \right] \quad (2.22)
\end{aligned}$$

- Only the tradable instrument  $U$ , and the numeraire  $Y$  (and their respective derivatives) and the "volatility"  $b(S, t)$  of the underlying process  $S$  appear in this expression.
- The real underlying drift  $a(S, t)$  has disappeared completely.

- Since the prices of financial instruments must be independent of the method used to compute them, the following theorem holds independent of the choice of numeraire:

**Theorem 5** *Suppose there exists a numeraire  $Y$  in an arbitrage-free market satisfying Equation 2.19 with a previsible process  $m(Y, t)$ . Then the drift of the underlying in the real world is irrelevant to the price of the financial instrument. Arbitrage-freedom alone determines the prices of financial instruments and not the expectation of the market with respect to the evolution of the underlying.*

- Now consider the behavior of the *non*-normed process of the tradable instrument  $U$  in the martingale measure.

- The real world process 2.2 is transformed via 2.21 to

$$\begin{aligned}
dU &= a_U dt + \frac{\partial U}{\partial S} b dW \\
&= a_U dt + \frac{\partial U}{\partial S} b \left\{ d\widetilde{W} - \frac{1}{b} \left[ a_U \left( \frac{\partial U}{\partial S} \right)^{-1} - \frac{d \ln Y}{dt} \left( \frac{\partial \ln U}{\partial S} \right)^{-1} \right] dt \right\} \\
&= a_U dt + \frac{\partial U}{\partial S} b d\widetilde{W} - \left[ a_U - U \frac{d \ln Y}{dt} \right] dt
\end{aligned}$$

- and thus

$$dU(S, t) = \frac{d \ln Y(t)}{dt} U(S, t) dt + b(S, t) \frac{\partial U(S, t)}{\partial S} d\widetilde{W} \quad (2.23)$$

- The drift of a tradable instrument in the martingale measure for a numeraire of the form 2.19 is simply the product of the price of the instrument and the yield of the numeraire!

**Theorem 6** *The expectation of the yield (defined as the expected relative price change per time) of a tradable financial instrument in the martingale measure of a numeraire*

of the form 2.19 is always equal to the yield of the numeraire instrument.

$$E_t^Y \left[ \frac{dU(S,t)}{U(S,t)} \right] = \frac{d \ln Y(t)}{dt} dt \quad (2.24)$$

- – These properties naturally hold for all tradable instrument since all tradable instruments are martingales with respect to the same probability measure.

## 2.3 The Market Price of Risk

- For every instrument  $d\widetilde{W}$  in Equation 2.7 is *the same* Brownian motion in the martingale measure.
- The Brownian motion  $dW$  of the underlying in the real world is not dependent on a specific financial instrument either.
- Thus, the change in drift from  $dW$  to  $d\widetilde{W}$  in 2.7 must be *the same* for every tradable instrument  $U$ .

– i.e. for two arbitrary tradable instruments,  $U_1(S, t)$  and  $U_2(S, t)$ :

$$\frac{a_{u_1} + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U_1 - b \frac{n}{Y} \frac{\partial U_1}{\partial S}}{b \frac{\partial U_1}{\partial S} - n \frac{U_1}{Y}} = \frac{a_{u_2} + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U_2 - b \frac{n}{Y} \frac{\partial U_2}{\partial S}}{b \frac{\partial U_2}{\partial S} - n \frac{U_2}{Y}} \quad (2.25)$$

– with the  $a_{u_i}$  *real* world drifts as in 2.2.

- Regardless of the appearance of the real world drift  $a_u$  of a financial instrument  $U$ , the combination in 2.25 must be the same for *all* financial instruments.
- This combination has its own name; it is called <sup>6</sup> *market price of risk*.

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<sup>6</sup>The motivation for this name will become clear further below, when we look at certain special cases.

- The market price of risk  $\gamma_U$  for an instrument  $U$  is defined by

$$\begin{aligned}\gamma_U(t) &:= \frac{a_u + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] U - b \frac{n}{Y} \frac{\partial U}{\partial S}}{b \frac{\partial U}{\partial S} - n \frac{U}{Y}} \\ &= \frac{1}{b} \left[ a_U \left( \frac{\partial U}{\partial S} \right)^{-1} - \frac{d \ln Y}{dt} \left( \frac{\partial \ln U}{\partial S} \right)^{-1} \right]\end{aligned}\tag{2.26}$$

- with  $a_u$  as in Equation 2.2.
- The second equality holds for a numeraire of the form 2.19.

**Theorem 7** *The market price of risk is identical for all tradable instruments in a complete, arbitrage-free market.*

Comparing the definition in Equation 2.26 with Equation 2.7 gives immediately that

**Theorem 8** *The previsible process  $\gamma(t)$  in the Girsanov theorem effecting the drift transformation for the transition from the probability measure in the real world into the martingale measure is the market price of risk.*

- The real world drift  $a_U$  of  $U$  can by the definition 2.26 be expressed in terms of the market price of risk:

$$a_U = \gamma_U b \frac{\partial U}{\partial S} + U \frac{d \ln Y}{dt}$$

- Substituting this into 2.2 yields the *real* world process for a  $U$ :

$$dU(S, t) = \left[ \gamma_U(t) b(s, t) \frac{\partial U(S, t)}{\partial S} + \frac{d \ln Y(t)}{dt} U(S, t) \right] dt + b(S, t) \frac{\partial U(S, t)}{\partial S} dW \quad (2.27)$$

- Comparing this with 2.23, the process for  $U$  in the martingal measure yields the following "cook book" recipe:

**Theorem 9** *Setting the market price of risk equal to zero in the the stochastic process (in the differential equation) which governs the financial instrument in the real world, one immediately obtains the stochastic process (i.e. the differential equation) which is to be applied in the valuation of this instrument.*

## 2.4 Tradable Underlyings

- Above equations appear relatively complicated because we have not assumed that the underlying  $S$  is tradable.
- If the underlying is tradable, it can be used directly in place of the instrument  $U$ :

$$U = S \implies \frac{\partial U}{\partial S} = 1 \quad , \quad \frac{\partial^2 U}{\partial S^2} = 0 = \frac{\partial U}{\partial t} = 0 \quad (2.28)$$

- Then the process 2.18 for  $S$  reduces to

$$dS = \frac{b \frac{n}{Y} - a \frac{n}{b} \frac{S}{Y} - \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] S}{1 - \frac{n}{b} \frac{S}{Y}} dt + b d\widetilde{W}$$

- The corresponding process, 2.22, for a more useful choice of numeraire satisfying 2.19 reduces to

$$dS(t) = S(t) \frac{d \ln Y(t)}{dt} dt + b(S, t) d\widetilde{W} \quad (2.29)$$

The expectation of the yield (defined as the expected relative price change per time) of a tradable underlying in the martingale measure is always equal to the yield of the numeraire instrument.

- The market price of risk  $\gamma_S(t)$  for a tradable underlying is obtained from 2.26 with 2.28:

$$\gamma_S(t) = \frac{a + \left[ \frac{n^2}{Y^2} - \frac{m}{Y} \right] S - b \frac{n}{Y}}{b - n \frac{S}{Y}} = \frac{a(S, t)}{b(S, t)} - \frac{S(t)}{b(S, t)} \frac{d \ln Y}{dt} \quad (2.30)$$

- $a = a(S, t)$  is the underlying drift in the real world.
- the second equality holds for a numeraire of the form 2.19.

- The process 2.1 in the *real* world thus becomes

$$dS(t) = \left[ b(S, t) \gamma_S(t) + \frac{d \ln Y(t)}{dt} S(t) \right] dt + b(S, t) dW \quad (2.31)$$

- Comparison with 2.29 again yields the cook book recipe: The market price of risk must be set equal to zero for pricing.

- This still holds for a very general case: It has only been assumed that the underlying is tradable and that the numeraire instrument is of the form 2.19.

## 2.5 Applications in the Black-Scholes World

- Consider now a very special case:
  - The underlying is tradable
  - The underlying process in the real world is

$$d \ln S(t) = \mu dt + \sigma dW$$

\* with constant volatility  $\sigma$ .

- The numeraire is a zerobond (maturing at some arbitrary future time  $T$ ):  
 $Y(t) = B(t, T)$
- Interest rates are constant.
- In short: the Black-Scholes world.

- The real world process for  $B$  is of the form 2.19, for continuous compounding explicitly:

$$dY \equiv dB(t, T) = \frac{dB(t, T)}{dt} dt = rB(t, T) dt \implies \frac{d \ln Y(t)}{dt} = r \quad (2.32)$$

- The process for  $S$  follows via Ito's Lemma (assignment):

$$dS(t) = \tilde{\mu}S(t) dt + \sigma S(t) dW \quad \text{with } \tilde{\mu} = \mu + \frac{\sigma^2}{2} \quad (2.33)$$

- This corresponds to the general process 2.1 with the parameters

$$a(S, t) = \tilde{\mu}S(t) = \left( \mu + \frac{\sigma^2}{2} \right) S(t)$$

$$b(S, t) = \sigma S(t)$$

- Since the underlying is tradable, Equation 2.29 directly yields the process to be used in the valuation:

$$dS(t) = rS(t) dt + \sigma S(t) d\tilde{W}$$

– Comparison with Equation 2.33 shows that the choice

$$\tilde{\mu} = r \quad \text{or equivalently} \quad \mu = r - \sigma^2/2 \quad (2.34)$$

for the drift transforms the real world process directly to the process to be used for pricing.

- The market price of risk of the underlying is simply

$$\gamma_S(t) = \frac{a(S, t) - rS(t)}{b(S, t)} = \frac{\tilde{\mu} - r}{\sigma} \quad (2.35)$$

This is zero for  $\tilde{\mu} = r$ . (In fact "setting the market price of risk equal to zero" is equivalent to "choosing the correct drift for pricing").

- 2.35 provides the motivation for the name "market price of risk".

- The underlying drift in the real world is  $a(S, t) = \tilde{\mu}S(t)$ .
- The expected underlying-*yield* is thus  $a(S, t)/S(t) = \tilde{\mu}$ .

- $\tilde{\mu} - r$  represents the excess yield above the risk-free rate, expected of the underlying in the real world.
  - The volatility  $\sigma$  can be viewed as a measure of the risk of the underlying.
  - Thus the  $\gamma_S$  can be interpreted as the yield in excess above the risk-free rate *per risk unit*  $\sigma$  which the market expects of the underlying.
  - This is, so to speak, the price the market requests for the risk of investing in the underlying.
- Finally we will now show that one can actually calculate something ....
  - The payoff of a path-independent instrument depends solely on the value of the underlying at maturity  $T$ :

$$V(S, T) = V(S(T), T)$$

- Therefore we only need the distribution of  $S$  at time  $T$  (and not the distribution of all paths of  $S$  between  $t$  and  $T$ ).

- Choosing the zero bond as the numeraire,  $Y(t) = B(t, T)$ , and using  $B(T, T) = 1$ , Equation 2.16 for the price  $V$  of a financial instrument becomes

$$V(S, t) = Y(t) \mathbf{E}_t^Y \left[ \frac{V(S, T)}{Y(T)} \right] = B(t, T) \mathbf{E}_t^Y [V(S(T), T)]$$

- We model the underlying with the simple process 2.33.
- The associated underlying process over a *finite* time interval of length  $\delta t = T - t$  is the solution of the stochastic differential equation 2.33 (Assignment!):

$$\begin{aligned} S(T) &= S(t) \exp [\mu(T - t) + \sigma W_{T-t}] \quad \text{with } W_{T-t} \sim \text{N}(0, T - t) \\ &= S(t) \exp [x] \quad \text{with } x \sim \text{N}(\mu(T - t), \sigma^2(T - t)) \end{aligned} \quad (2.36)$$

- The distribution of  $S(T)$  is therefore  $S(t)$  times the exponential of a normally distributed random variable.
- Thus the expectation of the function

$$V(S(T), T) = V(S(t)e^x, T) =: g(x)$$

is explicitly:

$$\mathbf{E}_t [V(S(T), T)] =: \mathbf{E}_t [g(x)] = \int_{-\infty}^{\infty} g(x)p(x)dx$$

– with  $p$  being the probability density of the normal distribution, i.e.

$$\mathbf{E}_t [V(S(T))] = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} V(S(t)e^x, T) \exp \left[ -\frac{(x - \mu(T-t))^2}{2\sigma^2(T-t)} \right] dx$$

- This is the expectation of the payoff with respect to a probability measure associated with  $\mu$ .
- To use this for the *valuation* of the instrument  $V$  it must be computed with respect to the (in a complete market, unique) *martingale* measure.
  - This is done by choosing the drift as in 2.34.

– The price of  $V$  then becomes

$$\begin{aligned} V(S, t) &= B(t, T) \mathbb{E}_t^Y [V(S(T), T)] \\ &= \frac{B(t, T)}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} V(S(t)e^x, T) \exp \left[ -\frac{\left[ x - \left( r - \frac{1}{2}\sigma^2 \right) (T-t) \right]^2}{2\sigma^2(T-t)} \right] dx \end{aligned} \tag{2.37}$$

- This equation holds in complete generality for *every* non-path dependent instrument with payoff  $V(S(T), T)$  on an underlying  $S$  of the form 2.33!
- To be more specific compute a plain vanilla call as an assignment.

# 3

## Interest Rates and Term Structure Models

- The assumptions in the Black-76 model (non-stochastic interest rates) contradict the very existence of interest rate options.
  - for deterministic interest rates, one would know at time  $t$  which options will

be in or out of the money upon maturity,  $T$ .

- \* out of the money options would be worthless
- \* in the money options would be nothing other than forward transactions.

- Nonetheless, option prices obtained by applying the Black-76 model are surprisingly good<sup>1</sup>.
- Allow for *stochastic* interest rates via so-called *term structure models*.
- An interest rate  $R(t, T)$  depends on *two* time variables
  - time  $t$  at which the interest rate is being considered
  - maturity  $T$  of the term over which the interest is to be paid
- For a fixed time  $t$  (today), the set of all interest rates for the various maturity dates  $T$  form the so-called *term structure*.

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<sup>1</sup>The Black-76 model can be derived as a special case of the Heath-Jarrow-Morton model.

- each point in this curve is a stochastic variable. The term structure is thus a continuum of infinitely many stochastic variables.
- These variables are naturally very strongly correlated; the interest rate over a term of 3 years and 2.34 seconds is (almost) the same as that over a term of 3 years and 0 seconds and so on.
- In practice, one considers only finitely many terms ( $\tau = T - t$  of 1 day, 1 month, 3 months, 6 months, 9 months, 1 year, 2 years, 3 years, 5 years, 10 years, etc.).
- Most term structure models even go a step further and reduce the number of factors driving the stochastic evolution of the entire term structure to 1 or 2 (in rare cases 3) stochastic variables.
  - One refers to these models respectively as 1-factor, 2-factor and 3-factor models.
  - Motivation for this: Principle component analysis shows, that usually far more than 90% of the dynamics of the yield curve can be explained by only one or two stochastic factors,

### 3.1 Instantaneous Spot Rates and Instantaneous Forward Rates

- A stochastic process is usually assumed for very short term rates, so-called *instantaneous* interest rates<sup>2</sup>.
  - either for the *instantaneous short rate* (also called the *instantaneous spot rate*)
  - or for the *instantaneous forward rate*.
  - For the terms  $\tau = T - t$  of those rates we consider the limit  $\tau \rightarrow 0$ , or equivalently  $T \rightarrow t$ .

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<sup>2</sup>In more recent models, the so-called *market rate models* or *Brace-Gatarek-Musiela models*, (*BGM-models*), this is no longer the case. For such models, an interest rate over a longer term (for example, the 3-month LIBOR) is used as the underlying stochastic process [4].

- The *instantaneous spot rate*  $r(t)$  is defined by

$$e^{-r(t)dt} := \lim_{dt \rightarrow 0} B(t, t + dt)$$

$$r(t) = -\lim_{dt \rightarrow 0} \frac{\ln B(t, t + dt)}{dt} \quad (3.1)$$

- The *instantaneous forward rate*  $f(t, T)$  is defined by

$$e^{-f(t, T)dT} := \lim_{dT \rightarrow 0} B(T, T + dT | t) = \lim_{dT \rightarrow 0} \frac{B(t, T + dT)}{B(t, T)}$$

$$f(t, T) = -\lim_{dT \rightarrow 0} \frac{1}{dT} \ln \frac{B(t, T + dT)}{B(t, T)} = -\lim_{dT \rightarrow 0} \frac{\ln B(t, T + dT) - \ln B(t, T)}{dT}$$

– Thus

$$f(t, T) = -\frac{\partial \ln B(t, T)}{\partial T} \quad (3.2)$$

- Integrating this equation over  $dT$  and making use of the fact that  $B(t, t) = 1$

yields

$$\int_t^T f(t, s) ds = - \int_t^T \frac{\partial \ln B(t, s)}{\partial s} ds = - \ln B(t, T) + \ln \underbrace{B(t, t)}_1 = - \ln B(t, T)$$

– and therefore

$$B(t, T) = \exp \left[ - \int_t^T f(t, s) ds \right] \quad (3.3)$$

- The forward rate over a *finite* time interval of length  $T' - T$  is thus the *average* of the instantaneous forward rates over this interval:

$$R(T, T' | t) = - \frac{1}{T' - T} \ln \frac{B(t, T')}{B(t, T)} = \frac{1}{T' - T} \int_T^{T'} f(t, s) ds \quad (3.4)$$

- relation between the *instantaneous* forward rates and *finite* zero bond yields (spot rates over finite time intervals of length  $T - t$ ):

$$f(t, T) = - \frac{\partial (\ln \exp [-R(t, T)(T - t)])}{\partial T} = \frac{\partial [R(t, T)(T - t)]}{\partial T}$$

(by inserting  $B(t, T)$  for continuous compounding into 3.2)

– after application of the product rule:

$$f(t, T) = R(t, T) + (T - t) \frac{\partial R(t, T)}{\partial T} \quad (3.5)$$

- The forward rates are greater than the spot rates for  $\partial R(t, T)/\partial T > 0$ , i.e. for term structures whose values increase with  $T$ .<sup>3</sup>
- Taking the limit  $T \rightarrow t$  yields the relationship between the instantaneous forward rate and the *instantaneous* spot rate:

$$r(t) = \lim_{T \rightarrow t} f(t, T) \quad (3.6)$$

- In anticipation of the following we stress that all of the above holds in any arbitrary probability measure, i.e. independently of any choice of numeraire, since everything followed directly from the definitions of instantaneous interest rates.

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<sup>3</sup>As a reminder: The term structure is per definition the spot rate curve  $R(t, T)$  as a function of maturity  $T$ .

## 3.2 Important Numeraire Instruments

- The value  $V$  of an arbitrary interest rate instrument<sup>4</sup> normalized with an arbitrary, tradable financial instrument  $Y$  is a process  $Z = V/Y$  which, according to Equation 1.17, is a martingale

$$Z(t) = \mathbb{E}_t^Y [Z(u)] \quad \forall u \geq t$$

- The martingale measure at time  $t$  depends on the choice of the numeraire instrument  $Y$ .
- The tradable instrument used as *numeraire* is, in principle, arbitrary.
- An appropriate choice of numeraire, however, may yield to elegant solutions of specific problems.
  - Similar to a well chosen coordinate system for problems in physics.
  - Two numeraire instruments are particularly popular.

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<sup>4</sup>In spot rate models, every interest rate instrument can be interpreted as a derivative  $V$  on the underlying  $S(t) = r(t)$  or  $S(t) = \ln r(t)$ .

### 3.2.1 The Risk-neutral Measure

- Define  $A(t_0, t)$  as the value of a so-called *bank account* or *money market account*.
  - Value of one monetary unit (for example, 1 euro) at time  $t$ , which was invested at a time  $t_0 < t$  and was subsequently always compounded at the current spot rate, with the interest earnings being immediately re-invested in the same account at the same rate.
  - Intuitively: reinvest ever decreasing interest payments earned over ever shorter interest periods over a finite time span, the number of interest periods finally approaching infinity.
- value of such an account written in terms of the instantaneous spot rate:

$$A(t_0, t) = \exp \left[ \int_{t_0}^t r(s) ds \right] \quad (3.7)$$

- The so-called *risk-neutral measure* uses the bank account, Equation 3.7, as the numeraire.

$$Y(t) = A(t_0, t) = \exp \left[ \int_{t_0}^t r(s) ds \right] \quad \text{for arbitrary } t_0 < t$$

- This numeraire has the important property, Equation 2.19:

$$\begin{aligned} Y(t + dt) &= A(t_0, t + dt) \\ &= \exp \left[ \int_{t_0}^{t+dt} r(s) ds \right] = \exp \left[ r(t)dt + \int_{t_0}^t r(s) ds \right] \\ &= e^{r(t)dt} \exp \left[ \int_{t_0}^t r(s) ds \right] = e^{r(t)dt} Y(t) \end{aligned}$$

– Therefore

$$dY(t) = Y(t + dt) - Y(t) = (e^{r(t)dt} - 1) Y(t) \approx r(t)Y(t)dt$$

- \* in the last step we used the approximation  $e^x \approx 1 + x$ . For infinitesimal  $x$  (i.e. infinitesimal  $dt$ ) equality holds.
- \*  $r(t)$  and  $Y(t)$  are known at time  $t$  (even if they are stochastic variables and as such not yet known for the next time step  $t + dt$ ).
- \* Therefore the process  $m(t) = r(t)Y(t)$  is previsible as required in Equation 2.19.

- With this numeraire the martingale property Equation 1.17 becomes

$$\frac{V(t)}{A(t_0, t)} = \mathbf{E}_t^A \left[ \frac{V(u)}{A(t_0, u)} \right] \quad \forall u \geq t \geq t_0$$

- The initial time point  $t_0$  in the money account is arbitrary.
  - For every  $t_0$  one obtains another, different risk-neutral measure.
  - Choosing  $t_0 = t$  and using  $A(t, t) = 1$  directly yields the price of the financial instrument:

$$V(t) = \mathbf{E}_t^A \left[ \frac{V(u)}{A(t, u)} \right] = \mathbf{E}_t^A \left[ e^{-\int_t^u r(s) ds} V(u) \right] \quad \forall u \geq t \quad (3.8)$$

**Today's value of a financial instrument is equal to the expectation of the discounted future value, taken with respect to the risk-neutral measure.**

- This is *not* the discounted future expectation!
  - In this measure, the discounting is performed *first* and *then* the expectation is calculated.

- Example: the value at time  $t$  of a zero bond with maturity  $u$ 
  - Set  $V(t) = B(t, u)$  and consequently  $V(u) = B(u, u) = 1$ , in Equation 3.8 to obtain

$$B(t, u) = \mathbf{E}_t^A \left[ \frac{1}{A(t, u)} \right] = \mathbf{E}_t^A \left[ e^{-\int_t^u r(s) ds} \right] \quad (3.9)$$

- In the risk neutral measure the bond price is the expectation of the reciprocal of the bank account.
- Comparing with Equation 3.3, which always holds, yields the relationship between the instantaneous forward rates and the future instantaneous spot rates in the risk-neutral measure.

$$e^{-\int_t^u f(t, s) ds} = \mathbf{E}_t^A \left[ e^{-\int_t^u r(s) ds} \right]$$

- The *future price*  $V(u)$  for  $u > t$  (unknown at time  $t$ ) must be distinguished from the *forward price*  $V(t, u)$  (known at time  $t$  from arbitrage arguments). If there are no dividends between  $t$  and  $u$  the forward price is

$$V(t, u) = \frac{V(t)}{B(t, u)} \quad \text{mit} \quad u > t \quad (3.10)$$

- Thus the forward price of an interest rate instrument in the risk-neutral measure is

$$\frac{V(t)}{B(t, u)} = \frac{1}{B(t, u)} \mathbf{E}_t^A \left[ \frac{V(u)}{A(t, u)} \right] = \mathbf{E}_t^A \left[ e^{-\int_t^u (r(s) - f(t, s)) ds} V(u) \right] \quad \forall u \geq t \quad (3.11)$$

### 3.2.2 The Forward-neutral Measure

- For the so-called *forward-neutral measure*, a zero bond is used as the numeraire (see Equation.3.3)

$$Y(t) = B(t, T) = \exp \left[ - \int_t^T f(t, s) ds \right] \quad \text{for arbitrary } T > t$$

- This numeraire also has the important property, Equation 2.19:

$$\begin{aligned}
 Y(t + dt) &= B(t + dt, T) = \exp \left[ - \int_{t+dt}^T f(t, s) ds \right] \\
 &= \exp \left[ - \int_t^T f(t, s) ds + \underbrace{f(t, t + dt) dt}_{r(t) \text{ for } dt \rightarrow 0} \right] \\
 &= e^{r(t)dt} \exp \left[ - \int_t^T f(t, s) ds \right] = e^{r(t)dt} Y(t)
 \end{aligned}$$

– thus

$$dY(t) = Y(t + dt) - Y(t) = (e^{r(t)dt} - 1) Y(t) \approx r(t)Y(t)dt$$

- \* The Taylor series expansion of the exponential function  $e^x \approx 1 + x$  was used here again. Equality holds for infinitesimal  $x$  (i.e. for infinitesimal  $dt$ ).
- \*  $r(t)$  and  $Y(t)$  are known at time  $t$  implying the previsibility of  $m(t) = r(t)Y(t)$  as required in Equation 2.19.

- With this numeraire, the martingale property equation 1.17 becomes

$$\frac{V(t)}{B(t, T)} = \mathbf{E}_t^B \left[ \frac{V(u)}{B(u, T)} \right] \quad \forall T \geq u \geq t$$

- The maturity  $T$  of the zero bond is arbitrary.
  - For each choice of  $T$ , one obtains a different measure.
  - Choosing  $T = u$  and using  $B(u, u) = 1$  directly yields the price of a financial instrument

$$V(t) = B(t, u) \mathbf{E}_t^B [V(u)] = e^{-\int_t^u f(t, s) ds} \mathbf{E}_t^B [V(u)] \quad \forall u \geq t \quad (3.12)$$

**Today's value of a financial instrument is equal to the discounted expectation of its future value, taken with respect to the forward neutral measure.**

- In this measure the expectation is taken *first* and *then* discounted.

- The forward price, Equation 3.10, in this measure exactly equals the expected future price (hence the name "forward-neutral"):

$$\frac{V(t)}{B(t, u)} = \mathbb{E}_t^B [V(u)] \quad \forall u \geq t \quad (3.13)$$

### 3.3 Deterministic Interest Rates

- Confused ?
  - Prices of financial instruments "usually" calculated by discounting the future expectation with  $B(t, T)$
  - i.e. in the *forward*-neutral measure.
  - However, everyone always speaks of the *risk*-neutral measure.
- In those instances however, interest rates are always assumed to be *deterministic* (or even constant).

- For deterministic interest rates, the forward-neutral and the risk-neutral measures are identical.
- Consider a portfolio consisting of
  - a zero bond  $B(t, T)$
  - a floating rate loan made at time  $t$  to exactly finance the purchase of the zero bond.
  - This portfolio has no value at time  $t$  and must have no value for all later times as well, since *no risk* is involved (everything is known at time  $t$ ).
- After one period  $\delta t$ 
  - the loan debt will have grown to  $B(t, T) \exp(r(t)\delta t)$
  - the value of the zero-bond must be equal to the value of the loan:

$$B(t + \delta t, T) = B(t, T) \exp(r(t)\delta t)$$

- So far, everything is as it was as for stochastic interest rates.

- On the right hand side there appear only terms which are known at time  $t$ .
- The difference comes in the next time step, i.e. at time  $t + 2\delta t$ 
  - the credit debt will have grown to  $B(t, T) \exp(r(t)\delta t) \exp(r(t + \delta t)\delta t)$  with a *known* interest rate  $r(t + \delta t)$
  - Since *no risk* is involved this must again equal the value of the zero bond at time  $t + 2\delta t$ .
- Proceeding analogously over  $n$  time steps the zero bond has value

$$B(t + n\delta t, T) = B(t, T) \exp \left[ \sum_{i=0}^{n-1} (r(t + i\delta t)\delta t) \right]$$

- Taking the limit as  $\delta t \rightarrow 0$ , the value of the bond at time  $u := t + n\delta t$  is

$$B(u, T) = B(t, T) \exp \left[ \int_t^u r(s) ds \right] \quad \text{for } u \geq t$$

- This holds for every  $u \geq t$ , in particular for  $u = T$ . Thus, observing that  $B(T, T) = 1$ , one obtains the price of a zero bond for deterministic interest rates:

$$B(t, T) = \exp \left[ - \int_t^T r(s) ds \right] = \frac{1}{A(t, T)} \quad (3.14)$$

- Hence, for deterministic interest rates, the forward-neutral numeraire is equal to the reciprocal of the risk-neutral numeraire. Substituting this into Equation 3.8, one obtains

$$V(t) = \mathbf{E}_t^A \left[ \frac{V(u)}{A(t, u)} \right] = \mathbf{E}_t^A [B(t, u)V(u)] = B(t, u)\mathbf{E}_t^A [V(u)]$$

- – Since  $B(t, u)$  is known at time  $t$  it has been factored out of the expectation.
- The price of a financial instrument must be independent of the measure used in its computation.

- In consequence, comparison with Equation 3.12 immediately yields

$$B(t, u)E_t^B [V(u)] = V(t) = B(t, u)E_t^A [V(u)]$$

- and thus:  $E_t^B [V(u)] = E_t^A [V(u)]$ . Both measures are identical if interest rates are deterministic!
- The fundamental difference to stochastic interest rates is that the price of a zero-bond in Equation 3.14 now depends on the *future spot rates* (In the general equation 3.3 the price of a zero bond is given by the *current forward rates*).

- The derivative of Equation 3.14 with respect to  $T$  gives

$$r(T) = -\frac{\partial}{\partial T} \ln B(t, T)$$

- Comparison with Equation 3.2 (which always holds) yields

$$f(t, T) = r(T) \quad \forall t, T \quad \text{for } t \leq T$$

- If interest rates are deterministic, the current instantaneous forward rates are equal to the (known) future instantaneous spot rates.

- Since the right hand side is not dependent on  $t$ , this must be true for the left hand side as well.
  - \* Hence, if interest rates are deterministic, the instantaneous forward rates are independent of the present time  $t$ .

### 3.4 Tradeable and Non-Tradeable Instruments

- Both the numeraire  $Y$  and the financial instrument  $V$  *must* be **tradeable** for the martingale property Equation 1.3 to hold.
- Otherwise one may not be able to exploit potential arbitrage opportunities.
- Examples for a non-tradeable variable: the *yield* of a tradeable instrument whose price is a non-linear function of its yield.
- Consider a zero bond  $B$  with lifetime  $\tau$ .
  - The yield  $r$  of the zero bond depends non-linearly on its price (except in strictly linear compounding convention).

– For example, for simple compounding:  $B = (1 + r\tau)^{-1}$

- The zero-bond is obviously a tradeable instrument.

– Therefore its future expectation taken in the forward-neutral measure as in Equation 3.13 *must* equal its forward price:

$$B(T, T + \tau | t) = \mathbb{E}_t^B [B(T, T + \tau)] = \mathbb{E}_t^B \left[ \frac{1}{1 + r(T)\tau} \right] \neq \frac{1}{1 + \mathbb{E}_t^B [r(T)]\tau}$$

– The inequality should emphasize that this is *not* equal to the price calculated with the expectation of the yield.

- On the other hand, the forward rate for the period between  $T$  and  $T + \tau$  is by definition equal to the yield of a forward zero bond over this period. For linear compounding explicitly:

$$\frac{1}{1 + r_f(T, T + \tau | t)\tau} \equiv B(T, T + \tau | t)$$

- Comparison with the above expression for the forward bond price gives

$$\begin{aligned}\frac{1}{1 + r_f(T, T + \tau | t)\tau} &\neq \frac{1}{1 + \mathbf{E}_t^B [r_\tau(T)]\tau} \\ &\implies \\ r_f(T, T + \tau | t) &\neq \mathbf{E}_t^B [r_\tau(T)]\end{aligned}$$

- The forward rate is thus *not* equal to the forward-neutral expectation of the future spot rate!
  - Because of Equation 3.12, the zero bond yields are therefore *not* martingales with respect to the forward-neutral measure.
  - But: *All tradeable* instruments are martingales with respect to the forward-neutral measure.
  - Hence: **Zero bond yields (i.e. interest rates) are not tradeable.**
- For example, the 3-month LIBOR as the yield of a 3-month zero bond is *not* tradeable, but the 3-month zero bond of course is.

- In *forward rate agreements*, *caplets* and *floorlets*, the difference between the future LIBOR and the strike is not paid out at the LIBOR-fixing at the beginning of the relevant interest period but rather is paid out at the *end* of that period.
- This has the effect that these instruments can be considered as forward contracts (or options) on future *zero bonds*, i.e. on *tradeable* instruments.
- This is the deeper reason why the Black76 valuation of FRAs and IRGs using forward rates works.

### 3.5 Convexity Adjustments

- By Equation 3.12, the price  $V(t)$  of a financial instrument is equal to its discounted future expectation in the forward-neutral measure .
  - Choose the future time to be the maturity  $T$  of the instrument.
  - Then the price is equal to the discounted future expectation of its *payoff profile*  $V(T)$ .

- If the payoff is a *linear* function of the underlying  $S$ , i.e.  $V(S, T) = a + bS$ , then "the expectation of the *payoff* is the payoff of the expectation":

$$\begin{aligned}
 \mathbb{E}[V(S, T)] &= \mathbb{E}[a + bS] \\
 &= \int_{-\infty}^{\infty} [a + bS] p(S) dS \\
 &= a \int_{-\infty}^{\infty} p(S) dS + b \int_{-\infty}^{\infty} S p(S) dS \\
 &= a + b\mathbb{E}[S]
 \end{aligned} \tag{3.15}$$

where  $p$  denotes the probability density of the martingale measure.

- Forward contracts have linear payoffs.
  - The payoff of a forward contract with maturity  $T$  and delivery price  $K$  is  $S(T) - K$ .
  - The expectation of this payoff is thus simply:

$$\mathbb{E}[V(S, T)] = \mathbb{E}[S(T) - K] = \mathbb{E}[S(T)] - K$$

- If the underlying  $S$  is itself a *tradeable* instrument (for example, a stock), one could now go a step further.
  - In the forward-neutral measure (or in the case of deterministic interest rates), the future expectation of each tradeable instrument is equal to its current forward price, i.e.  $E_t^B [S(T)] = S(t, T)$ .
  - Now using Equation 3.12 we get today's price of the forward contract:

$$\begin{aligned}
 V(S, t) &= B(t, T) E_t^B [V(S, T)] \\
 &= B(t, T) (E_t^B [S(T)] - K) \\
 &= B(t, T) (S(t, T) - K)
 \end{aligned}$$

- \* In the first line, Equation 3.12 was used
  - \* in the second line, Equation 3.15 (linear payoff)
  - \* in the third line, the namesake property of the forward-neutral measure, Equation 3.13 has been used.
- In some traditional pricing methods this last step is done even if the underlying is *not* a tradeable instrument! (and as such, not a martingale with respect to the forward-neutral measure).

- This mistake is then corrected (approximately) after the fact by means of a so-called *convexity adjustment*.
- The convexity adjustment is defined as the difference between the future expectation (in the forward-neutral measure) and the forward price

$$\text{Convexity Adjustment} \equiv \mathbb{E}_t^B [S(T)] - S(t, T) \quad (3.16)$$

- The most important case is when the underlying is an interest rate. This is not tradeable because the expression relating yields to prices of (tradeable) instruments (zero bonds), is non-linear. Explicitly:

$$B_r(t, t + \tau) = \begin{cases} \exp(-r\tau) & \text{continuous} \\ (1 + r)^{-\tau} & \text{discrete} \\ (1 + r\tau)^{-1} & \text{simple} \\ 1 - r\tau & \text{linear} \end{cases} \quad (3.17)$$

- We now determine an approximation for the convexity adjustment of the yield of such a zero bond, i.e. we determine:

$$\text{Convexity Adjustment} = \mathbb{E}_t^B [r(T, T + \tau)] - r_f(T, T + \tau | t)$$

- To find the expectation of the unknown future spot rate we expand the (also unknown) future bond price  $B_r(T, T + \tau)$  as a Taylor series up to second order about the (known) forward rate  $r_f = r(T, T + \tau | t)$ :

$$\begin{aligned}
 B_r(T, T + \tau) &= \sum_{n=0}^{\infty} \frac{[r(T, T + \tau) - r_f(T, T + \tau | t)]^n}{n!} \left. \frac{\partial^n B_r(T, T + \tau)}{\partial r^n} \right|_{r=r_f} \\
 &\approx B_{r_f} + [r - r_f] B'_r|_{r=r_f} + \frac{[r - r_f]^2}{2} B''_r|_{r=r_f}
 \end{aligned}$$

- We now calculate the forward-neutral expectation of this bond price<sup>5</sup>:

$$\underbrace{\mathbb{E}_t^B [B_r]}_{B_{r_f}} \approx B_{r_f} + B'_r|_{r=r_f} \mathbb{E}_t^B [r - r_f] + \frac{1}{2} B''_r|_{r=r_f} \underbrace{\mathbb{E}_t^B [(r - r_f)^2]}_{\approx \text{var}[r]} \quad (3.18)$$

- $B$  and its derivative are evaluated at  $r_f$  which is known as of today. Therefore,  $B$  and its derivative can be factored out of the expectation.

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<sup>5</sup>We have dropped all time arguments for ease of notation.

- The forward-neutral expectation of the bond price is exactly the forward bond price  $B_{r_f}$ .
- $E_t^B [(r - r_f)^2]$  is approximately equal to the variance of the future  $r$ 
  - \* it would be exactly this variance if  $r_f$  were equal to  $E_t^B [r]$ .
  - \* To express the variance of  $r$  in terms of values known at time  $t$ , the variance of  $r$  is approximated by the variance of the *forward rate*  $r_f$
  - \* both approximations together

$$E_t^B [(r - r_f)^2] \approx \text{var} [r] \approx \text{var} [r_f] = r_f^2 \sigma_f^2 (T - t) \quad (3.19)$$

- \* The volatility  $\sigma_f$  of the forward rate  $r_f$  is called *forward volatility*. This is (at least in principle) known at time  $t$ .
- Substituting all of the above into Equation 3.18 yields

$$0 \approx B_r'|_{r=r_f} (E_t^B [r] - r_f) + \frac{1}{2} B_r''|_{r=r_f} r_f^2 \sigma_f^2 (T - t)$$

- This can now be solved for the desired future expectation

$$E_t^B [r] \approx r_f - \underbrace{\frac{1}{2} r_f^2 \sigma_f^2 (T-t) \frac{B_r''|_{r=r_f}}{B_r'|_{r=r_f}}}_{\text{convexity adjustment}} \quad (3.20)$$

- Within all these approximations the Convexity Adjustment is explicitly for the four most widely used compounding conventions :

$$\begin{aligned} \text{Convexity Adjustment} &\approx -\frac{1}{2} r_f^2 \sigma_f^2 (T-t) \frac{B_r''|_{r=r_f}}{B_r'|_{r=r_f}} \quad (3.21) \\ &= \begin{cases} \frac{1}{2} r_f^2 \sigma_f^2 (T-t) \tau & \text{continuous} \\ \frac{1}{2} r_f^2 \sigma_f^2 (T-t) \frac{\tau(\tau+1)}{1+r_f} & \text{discrete} \\ r_f^2 \sigma_f^2 (T-t) \frac{\tau}{1+\tau r_f} & \text{simple} \\ 0 & \text{linear} \end{cases} \end{aligned}$$

- As expected: no convexity adjustment for strictly linear compounding.
  - \* In this compounding method, the interest rate is a linear combination of money and zero bonds, i.e. a portfolio of tradeable instruments and as such also tradeable.

- \* Thus, for the strictly linear compounding convention, the interest rate is itself a martingale and Equation 3.13 holds for the forward-neutral measure.

### 3.5.1 LIBOR-In-Arrears Swaps

- No problems regarding tradeability for ordinary FRAs, Swaps and IRGs since the interest rate (difference) fixed at the *beginning* of an interest period is paid at the *end* of the period.
  - Such contracts are effectively forward contracts on (tradeable) zero bonds.
- Swaps for which the difference between the future LIBOR and the fixed side is paid *at the same time* when the LIBOR rate is fixed are called *LIBOR-In-Arrears Swaps*.
  - Here the underlying is directly the LIBOR rate which is *not* tradeable.
  - If evaluated as if one could replace the future expectation of the underlying (with respect to the forward-neutral measure) with the forward rate, the

resulting error must be corrected, at least approximately, with the convexity adjustment.

- Consider only one period of a LIBOR-In-Arrears Swap, a *LIBOR-In-Arrears FRA*<sup>6</sup>.

- Principle  $N$ , fixed interest  $K$ , extends over a period from  $T$  to  $T + \tau$ .
- A potential compensation payment is calculated by simple compounding and flows directly when at time  $T$  when the LIBOR rate  $r(T, T + \tau)$  is fixed, i.e. the payoff is

$$V(r(T, T + \tau), T) = N\tau [r(T, T + \tau) - K] \quad (3.22)$$

- This is a *linear* function of the underlying  $r$ . The expectation (in any measure) of the payoff profile is thus simply

$$E[V(r(T, T + \tau), T)] = N\tau E[r(T, T + \tau)] - N\tau K \quad (3.23)$$

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<sup>6</sup>A LIBOR-In-Arrears Swap is simply a portfolio consisting of such FRAs.

– The value today, calculated with the forward-neutral measure, is

$$\begin{aligned} V(r(t), t) &= B(t, T) \mathbb{E} [V(r(T, T + \tau), T)] \\ &= B(t, T) N \tau \left( \mathbb{E}_t^B [r(T, T + \tau)] - K \right) \end{aligned}$$

– Up to now, all the equations have been exact.

- The future expectation is now determined either using a term structure model or (approximately) by means of the convexity adjustments given in Equation 3.20:

$$\begin{aligned} &V(r(T, T + \tau), t) \\ &= B(t, T) N \tau \left( \mathbb{E}_t^B [r(T, T + \tau)] - K \right) \\ &\approx B(t, T) N \tau \left( r_f(T, T + \tau | t) - \frac{1}{2} r_f(T, T + \tau | t)^2 \sigma_f^2 (T - t) \frac{B_r''(T, T + \tau)|_{r=r_f}}{B_r'(T, T + \tau)|_{r=r_f}} - K \right) \\ &= B(t, T) N \tau \left( r_f(T, T + \tau | t) + r_f(T, T + \tau | t)^2 \sigma_f^2 (T - t) \frac{\tau}{1 + \tau r_f} - K \right) \end{aligned} \tag{3.24}$$

– Simple compounding has been used in the last step.

- Thus for  $r_f$ , one must use the forward rate *with respect to simple compounding* as well.
- All of the above only payoffs which are *linear* functions of the underlying.
  - Only then does Equation 3.15 hold
  - Only then comes the expectation of the *underlying* into play.
- For *non-linear* payoffs, the expectation of the *payoff profile* must be calculated directly.
  - Example: plain vanilla

$$\begin{aligned}
 \mathbb{E}[\max\{S(T) - K, 0\}] &= \int_{-\infty}^{\infty} \max\{S - K, 0\} p(S) dS \\
 &= \int_K^{\infty} (S - K) p(S) dS \neq \max\{\mathbb{E}[S(T)] - K, 0\}
 \end{aligned}$$

### 3.5.2 Money Market Futures

- LIBOR-In-Arrears FRA may appear a bit academic.
- In reality: so-called *money market futures*
  - LIBOR-in Arrears FRAs traded on an exchange
  - among the most actively traded interest derivatives
- A money market future with a nominal  $N$ , a fixed rate  $K$  over a period from  $T$  to  $T + \tau$  yields (theoretically) at maturity  $T$  a compensation payment calculated with simple compounding as in Equation 3.22:

$$V(T) = N\tau [r(T, T + \tau) - K]$$

- Instrument has a *future styled* payment mode
  - The changes in the position's value are *immediately* realized on a margin account.

- Therefore today's value is directly equal to the future expectation, Equation 3.23, without discounting:

$$V(t) = N\tau (\mathbb{E}_t^B [r(T, T + \tau)] - K) \quad (3.25)$$

- This equals zero when  $K = \mathbb{E}[r(T, T + \tau)]$ .
  - When the contract is initiated the fixed rate  $K$  is always chosen in this way.
  - The fixed rate  $K$  of a money market futures contracted at time  $t$  thus gives directly the information on the opinion of the market on the *future expectation*  $\mathbb{E}[r(T, T + \tau)]$  of the interest rate.
  - Since the interest rate is not a tradeable instrument, this is *not* equal to the forward rate.
- Money market futures are heavily used in yield curve construction for terms between ca. 3 months and ca. 2 years.
  - To construct the current yield curve, the *forward* rates are needed (since from these, we can calculate the spot rates quite easily from arbitrage considerations)

- To determine the forward rate  $r_f(T, T + \tau | t)$  from quotes on money market futures, the convexity adjustment must be *subtracted* from the expectation as in Equation 3.16 to obtain

$$\begin{aligned}
 r_f(T, T + \tau | t) &= E_t^B [r(T, T + \tau)] - \text{Convexity Adjustment} \\
 &\approx E_t^B [r(T, T + \tau)] + \frac{1}{2} r_f(T, T + \tau | t)^2 \sigma_f^2 (T - t) \frac{B_r''(T, T + \tau)|_{r=r_f}}{B_r'(T, T + \tau)|_{r=r_f}} \\
 &= E_t^B [r(T, T + \tau)] - r_f(T, T + \tau | t)^2 \sigma_f^2 (T - t) \frac{\tau}{1 + \tau r_f}
 \end{aligned}$$

\* In the last step linear compounding was used.

- This is a non-linear equation which can be solved numerically for the unknown  $r_f$ .
- The convexity adjustments above is an approximation.
- The future expectation  $E_t^B [r(T, T + \tau)]$  can be calculated by other means, for example, using a term structure model.
  - One then obtains *another* expression for  $E_t^B [r(T, T + \tau)]$ .

- The forward rates are determined solely from the current spot rates and arbitrage considerations and are therefore independent of the model.
- Therefore the convexity adjustment is model dependent.
- For consistency the convexity adjustment used in constructing the current yield curves (from money market futures) should be consistent with the term structure model applied for pricing.

### Quotation for Money Market Futures

- Not the delivery price  $K = E_t^B [r(T, T + \tau)]$  is quoted but rather

$$\text{Quote}_{\text{Money Market Future}} = 100\% - E[r(T, T + \tau)]$$

- A quote of 96, 52%, for example, on a money market future means that, in the opinion of the market, the expectation (in the forward neutral measure) for the future 3-month rate is  $E[r(T, T + \tau)] = 3.48\%$ .
- The value at time  $t$  of a futures position contracted at time  $t = 0$  with  $K = E_0^B [r(T, T + \tau)]$

is given, according to Equation 3.25, by

$$\begin{aligned} V(t) &= N\tau (\mathbf{E}_t^B [r(T, T + \tau)] - \mathbf{E}_0^B [r(T, T + \tau)]) \\ &= N\tau \underbrace{(1 - \mathbf{E}_0^B [r(T, T + \tau)])}_{\text{Quote at Time } t=0} - N\tau \underbrace{(1 - \mathbf{E}_t^B [r(T, T + \tau)])}_{\text{Quote at Time } t} \end{aligned}$$

- A money market future on a 3-month LIBOR (i.e.  $\tau = 1/4$ ) with a nominal of  $N = 1,000,000$  which was agreed to at a price of 96.52% and which currently has a price of 95.95% is thus valued at

$$\begin{aligned} V(t) &= \frac{1,000,000 \text{ Euro}}{4} (96.52\% - 95.95\%) \\ &= 1425.00 \text{ Euro} \end{aligned}$$

- This amount is deposited in a margin amount.
- A change in the value of this position is reflected daily in the balance of the margin account.

## 3.6 Arbitrage-free Interest Rate Trees

- Start from the martingale property, Equation 1.17 (justifying the name arbitrage-free<sup>7</sup>).
- Select an appropriate numeraire.
- Discretize the integrals necessary for the expectations (with respect to the chosen measure) in a tree-structure.
- This procedure will be demonstrated explicitly in this lecture for 1-factor short rate models.
  - These models have the instantaneous short rate, defined in Equation 3.1, as their one and only stochastic driver.
- We will use the risk-neutral measure, i.e. the bank account as numeraire.
- The price of every instrument is then given by Equation 3.8.

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<sup>7</sup>So-called equilibrium models are not to be investigated here. Our treatment will be restricted to arbitrage-free pricing methods.

- Discretize first with respect to time by partitioning the time axis in intervals of length  $\delta t$ .

- $\delta t$  should be so small that the (stochastic) short rate is assumed to be a constant over  $\delta t$ :

$$\begin{aligned}
 V(t) &= \mathbf{E}_t^A \left[ e^{-\int_t^{t+\delta t} r(s) ds} V(t + \delta t) \right] & (3.26) \\
 &\approx \mathbf{E}_t^A \left[ e^{-r(t)\delta t} V(t + \delta t) \right] \\
 &= e^{-r(t)\delta t} \mathbf{E}_t^A [V(t + \delta t)] \\
 &= B(t, t + \delta t) \mathbf{E}_t^A [V(t + \delta t)]
 \end{aligned}$$

- In the last line the last equation, the risk-neutral price and the forward-neutral price (see Equation 3.12) can not be distinguished from one another over the very *short* time interval  $\delta t$ .

- \* This is consistent, since  $r$  is taken to be constant (in particular, deterministic) over this short time interval.
- \* This does not hold for longer time spans, since  $r$  changes randomly from one interval of length  $\delta t$  to the next.

- \* Thus, globally, we are in the risk-neutral measure even though the local equations may look "forward-neutral".
- A discount factor over a period of length  $\delta t$ , like for instance  $B(t, t + \delta t)$ , will be called *instantaneous discount factor* oder *instantaneous zero bond*.

### 3.6.1 Backward Induction

- Up to now we got the numeraire (the bank account) out of the expectation.
- It "only" remains to calculate the expectation.
- Discretize the continuous range of  $V$  after one time step of length  $\delta t$  into finitely many values.
  - Allowing two different values generates a binomial tree, three, a trinomial tree, etc.
  - The different values of  $V$  at time  $t + \delta t$  result directly from the different possible term structures. These in turn are determined by the values which can be assumed by the short rate.

- We will use binomial trees and assume that of a time step  $\delta t$  the short rate increases to the  $r_u$  or decreases to  $r_d$  with a probability  $p$  and  $1 - p$ , respectively.
- Several ways to enforce the martingale property:
  - fix the values  $r_u$  and  $r_d$  and adjust the probability  $p$  to achieve the arbitrage-free measure (this is done in the finite difference method where the grid is specified in advance),
  - fix the probability  $p$  and adjust the values  $r_u$  and  $r_d$  to achieve the arbitrage-free measure. We will do this and (arbitrarily) set

$$p = 1/2 \tag{3.27}$$

- In this model, i.e. in the binomial tree with  $p = 1/2$  the above expectation

becomes

$$\begin{aligned} V(t) &\approx B(t, t + \delta t) \mathbb{E}_t^A [V(t + \delta t)] \\ &\approx B(t, t + \delta t) [p V(r_u, t + \delta t) + (1 - p) V(r_d, t + \delta t)] \\ &= B(t, t + \delta t) \left[ \frac{1}{2} V(r_u, t + \delta t) + \frac{1}{2} V(r_d, t + \delta t) \right] \\ &= B(t, t + \delta t) \left[ \frac{1}{2} V_u + \frac{1}{2} V_d \right] \end{aligned} \tag{3.28}$$

– short form notation:  $V_u := V(r_u, t + \delta t)$ , etc.

• One time step later at time  $t + \delta t$ :

- $r_u \rightarrow r_{uu}$  with probability  $p$ ,  $r_u \rightarrow r_{ud}$  with probability  $1 - p$ , and similarly for  $r_d$ .
- In addition now two *different* discount factors  $B(t + \delta t, t + 2\delta t)$  at  $r_u$  and  $r_d$ .

–  $V_u$ , for example, reads

$$\begin{aligned} V_u &\approx B_{r_u}(t + \delta t, t + 2\delta t) \mathbb{E}_t^A [V_u(t + 2\delta t)] \\ &\approx B_{r_u}(t + \delta t, t + 2\delta t) \left[ \frac{1}{2}V(r_{uu}, t + 2\delta t) + \frac{1}{2}V(r_{ud}, t + 2\delta t) \right] \\ &= B_u \left[ \frac{1}{2}V_{uu} + \frac{1}{2}V_{ud} \right] \end{aligned}$$

\* with obvious short form notation  $V_{uu} := V(r_{uu}, t + 2\delta t)$ ,  $B_u := B_{r_u}(t + \delta t, t + 2\delta t)$ , etc.

– Analogously,

$$V_d \approx B_d \left[ \frac{1}{2}V_{du} + \frac{1}{2}V_{dd} \right]$$

– Substituting this into Equation 3.28 yields the price  $V(t)$  given by a two step binomial tree

$$V(t) \approx B(t, t + \delta t) \left[ \frac{1}{2}B_u \left[ \frac{1}{2}V_{uu} + \frac{1}{2}V_{ud} \right] + \frac{1}{2}B_d \left[ \frac{1}{2}V_{du} + \frac{1}{2}V_{dd} \right] \right] \quad (3.29)$$

- Analogously with  $V_{dd} \approx B_{dd} [\frac{1}{2}V_{ddu} + \frac{1}{2}V_{ddd}]$ , etc., after three steps the price is

$$\begin{aligned}
 V(t) \approx & B(t, t + \delta t) \left[ \frac{1}{2}B_u \left[ \frac{1}{2}B_{uu} \left[ \frac{1}{2}V_{uuu} + \frac{1}{2}V_{uud} \right] + \frac{1}{2}B_{ud} \left[ \frac{1}{2}V_{udu} + \frac{1}{2}V_{udd} \right] \right. \right. \\
 & \left. \left. + \frac{1}{2}B_d \left[ \frac{1}{2}B_{du} \left[ \frac{1}{2}V_{duu} + \frac{1}{2}V_{dud} \right] + \frac{1}{2}B_{dd} \left[ \frac{1}{2}V_{ddu} + \frac{1}{2}V_{ddd} \right] \right] \right] \quad (3.30)
 \end{aligned}$$

- And so on: Start at maturity and go backwards through the tree (*backward induction*) analogue to the treatment of options on stocks or exchange rates with binomial trees.
- The short rates  $r_u, r_d, r_{uud}$  (or equivalently the discount factors  $B_u, B_d, B_{uud}$ ), etc. must be chosen so that the *prices calculated* using the tree agree with the *market prices* of traded instruments.
- In particular, *zero bonds* (i.e. today's term structure) must be exactly reproduced.

- At each time point  $t+i\delta t$  there are  $2^i$  nodes in a (non-recombinant) binomial tree.
  - That's  $2^i$  unknowns, i.e. all permutations of  $u$  and  $d$  from  $r_{uu\dots u}$  (or  $B_{uu\dots u}$ ) to  $r_{dd\dots d}$  (or  $B_{uu\dots u}$ ).
  - The number of unknowns increases exponentially with the number of time steps!
  - So many conditions can not be generated by market prices of tradeable instruments.
  - In particular, no such functional relation between the number of instruments in the market and the number of time steps (depending only on the numerical implementation) in a tree exist.
  - Another condition must first be established preventing the exponential growth of the number of unknowns.
- We requiring that the tree be *recombinant*.
    - $r_{ud} = r_{du}$ ,  $r_{uud} = r_{udu} = r_{duu}$ , etc

- Ok for all financial instruments with path-independent payoff profiles, i.e. for  $V_{ud} = V_{du}$ ,  $V_{uud} = V_{udu} = V_{duu}$ , etc.
- Still good for path dependent instruments if tree is used as a basis for Monte Carlo (see later)
- A recombinant binomial tree has only  $i + 1$  nodes after  $i$  steps.

- Notation

- The node of a *recombinant* tree is uniquely determined by the ordered pair  $(i, j)$  (see Figure 3.1), where
  - \*  $i$  is the number of *up*-moves, starting from  $(0, 0)$
  - \*  $j$  is the number of *down*-moves, starting from  $(0, 0)$
  - \* For example,  $V(1, 2) = V_{udd} = V_{dud} = V_{ddu}$ , etc.

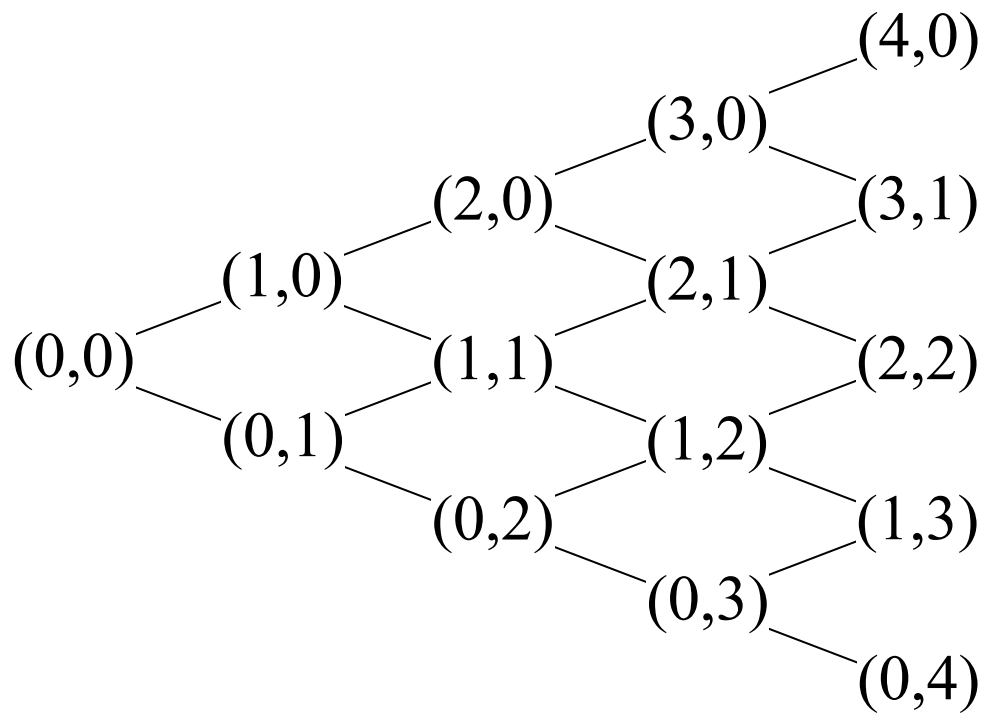


Figure 3.1: The binomial tree with the indexing showing the number of up- and down-moves required to get to the nodes starting from node  $(0,0)$ . For instance it takes 2 up-moves and 1 down-move to get to the node  $(2,1)$ .

- We use always this notation in the following, i.e.:

$$\begin{aligned}
 (i, j) &= \text{Node after } i \text{ up moves and } j \text{ down moves} \\
 r(i, j) &= \text{Instantaneous short rate at the node } (i, j) \\
 V(i, j) &= \text{Value of an interest rate instrument at the node } (i, j) \\
 B(i, j) &= \text{Value of instantaneous zerobond at the node } (i, j)
 \end{aligned} \tag{3.31}$$

- In particular,  $B(t, t + \delta t) = B(0, 0)$ .
- In this notation, the prices for (path independent) financial instruments after one binomial step is

$$\begin{aligned}
 V(t) &= V(0, 0) \\
 &\approx V(1, 0) \frac{1}{2} B(0, 0) \\
 &\quad + V(0, 1) \frac{1}{2} B(0, 0)
 \end{aligned} \tag{3.32}$$

- This holds not only at node  $(0, 0)$  but everywhere in the tree, yielding the **central rekursion relation for backward induction**:

$$V(i, j) \approx B(i, j) \left[ \frac{1}{2}V(i+1, j) + \frac{1}{2}V(i, j+1) \right] \quad (3.33)$$

- This equation applied to  $V(1, 0)$  and  $V(0, 1)$  yields the price after two binomial steps

$$\begin{aligned} V(t) &\approx V(2, 0) \frac{1}{4} B(0, 0) B(1, 0) \\ &\quad + V(1, 1) \frac{1}{4} B(0, 0) [B(1, 0) + B(0, 1)] \\ &\quad + V(0, 2) \frac{1}{4} B(0, 0) B(0, 1) \end{aligned} \quad (3.34)$$

– and after three

$$\begin{aligned} V(t) \approx & V(3,0) \frac{1}{8} B(0,0) B(1,0) B(2,0) \\ & + V(2,1) \frac{1}{8} B(0,0) [B(1,0) B(2,0) + B(1,1) [B(1,0) + B(0,1)]] \\ & + V(1,2) \frac{1}{8} B(0,0) [B(0,1) B(0,2) + B(1,1) [B(1,0) + B(0,1)]] \\ & + V(0,3) \frac{1}{8} B(0,0) B(0,1) B(0,2) \end{aligned} \tag{3.35}$$

an so on.

– The expressions were purposely written in terms of the prices at the last node in each branch.

### 3.6.2 Forward Induction and Greens-Functions

- We can't calculate anything yet!
- The instantaneous discount factors  $B(i, j)$  at the nodes  $(i, j) \neq (0, 0)$  are all unknown!

- Because we haven't yet constructed the tree for the short rate, i.e. for the instantaneous discount factors.
- This will be done via the so-called *forward induction*.
- But first: Introduce a class of extremely useful "artificial" instruments.
  - One such artificial instrument pays exactly one monetary unit if and only if the underlying (the short rate) attains the tree node  $(i, j)$  (this can only happen at time  $t + (i + j)\delta t$ ).
  - The value at time  $t$  of this instrument is denoted by  $G(i, j)$ .
  - $G(i, j)$  is the value at time  $t$ 
    - \* of one single monetary unit paid out
    - \* at one single node of the tree, namely at node  $(i, j)$ .
  - In this sense,  $G$  is the system's reaction to a perturbation of magnitude one at a single point.
    - \* This is analogous to the Green's functions in physics. We will therefore refer to  $G$  as a *Green's function*.

- By definition

$$G(0, 0) \equiv 1 \quad (3.36)$$

- For further values of the Green's function, set the value  $V(i, j)$  in Equations 3.32, 3.34 and 3.35 equal to one at exactly one node and zero on all the other nodes.

– Equation.3.32 then yields

$$G(1, 0) = \frac{1}{2}B(0, 0) = G(0, 1) \quad (3.37)$$

– Equation.3.34 yields

$$G(2, 0) = \frac{1}{4}B(0, 0) \quad B(1, 0) = \frac{1}{2}G(1, 0)B(1, 0)$$

$$G(0, 2) = \frac{1}{4}B(0, 0) \quad B(0, 1) = \frac{1}{2}G(0, 1)B(0, 1)$$

$$G(1, 1) = \frac{1}{4}B(0, 0) [B(1, 0) + B(0, 1)] = \frac{1}{2}G(1, 0)B(1, 0) + \frac{1}{2}G(0, 1)B(0, 1)$$

– and Equation.3.35 gives

$$G(3, 0) = \frac{1}{2}G(2, 0)B(2, 0)$$

$$G(0, 3) = \frac{1}{2}G(0, 2)B(0, 2)$$

$$G(2, 1) = \frac{1}{2}G(2, 0)B(2, 0) + \frac{1}{2}G(1, 1)B(1, 1)$$

$$G(1, 2) = \frac{1}{2}G(0, 2)B(0, 2) + \frac{1}{2}G(1, 1)B(1, 1)$$

– In general: A recursion relation (will be proven for an even more general case later)

$$G(i, j) = \frac{1}{2}G(i, j - 1)B(i, j - 1) + \frac{1}{2}G(i - 1, j)B(i - 1, j) \quad \text{for } i > 0, j > 0$$

$$G(i, 0) = \frac{1}{2}G(i - 1, 0)B(i - 1, 0)$$

$$G(0, j) = \frac{1}{2}G(0, j - 1)B(0, j - 1) \tag{3.38}$$

- All path independent instruments can be represented with Green's function
  - A payoff profile of the form  $f(r, T)$  can be distributed on the nodes  $(i, j)$  as appropriate:

$$f(r, T) \rightarrow f_T(i, j) := f(r(i, j), T) \quad \text{for } t + (i + j)\delta t = T \quad \forall i, j$$

- The value at time  $t$  of an individual payment  $f_T(i, j)$  at node  $(i, j)$  is of course just
  - \* the Green's function belonging to this node (which gives a value of exactly one monetary unit)
  - \* times the number of monetary units that are to be paid, i.e. times  $f_T(i, j)$ .
- The total value  $V(t)$  of the instrument is then simply

$$V(t, T) = \sum_{(i,j)} f_T(i, j) G(i, j) \quad \text{for } t + (i + j)\delta t = T \quad \forall i, j$$

- \* where  $\sum_{(i,j)}$  denotes "the sum over the nodes  $(i, j)$ ".

- Of course also true for payoffs defined on *arbitrary* nodes (which need not all lie in same time slice  $T$ )

$$V(t) = \sum_{(i,j)} f(i, j) G(i, j) \quad \text{for arbitrary payoff profiles } f(i, j) \quad (3.39)$$

– Therefore *path independence* is

- \* *not* the restriction that the payoff profile depend only on the time point  $T$ .
  - \* it may indeed depend on (the interest rate at) *all* possible nodes,
  - \* not however, on the path taken to arrive at these nodes (since this information is not available in the recombinant tree).
- A zero bond  $B(t, T)$  pays one monetary unit at time  $T$ , regardless of the state

of the underlying

$$\begin{aligned} B(t, T) &= \sum_{(i,j)} G(i, j) \times 1 \quad \text{for } t + (i + j)\delta t = T \\ &= \sum_{i=0}^n G(i, n - i) \quad \text{for } n = \frac{T - t}{\delta t} \end{aligned} \quad (3.40)$$

- This *theoretical* price has to exactly match the *market* price of the zero bond.
- The *market* prices  $B(t, T)$  for all zero bonds are available from the current term structure (constructed, for example, from traded benchmark bonds and some interpolations in between).
- In particular market prices for all zero bonds maturing at "grid point times"  $t + i\delta t$  are available for all  $i$ .
- Hence, one obtains one single no-arbitrage-condition for each time point  $t + i\delta t$ .

– For example, for  $i = 1$ ,

$$\begin{aligned} B(t, t + 1\delta t) &= \sum_{(i,j)} G(i, j) \quad \text{with } i + j = 1 \\ &= G(1, 0) + G(0, 1) \\ &= B(0, 0) \end{aligned}$$

\* where the Green's function as in Equation 3.37 has been used.

\* That was only a simple check for consistency.

– More interesting at the next maturity date:

$$\begin{aligned} B(t, t + 2\delta t) &= \sum_{(i,j)} G(i, j) \quad \text{with } i + j = 2 \\ &= G(2, 0) + G(1, 1) + G(0, 2) \\ &= \frac{1}{2}G(1, 0)B(1, 0) + \frac{1}{2}G(1, 0)B(1, 0) + \frac{1}{2}G(0, 1)B(1, 0) + \frac{1}{2}G(0, 1)B(1, 0) \\ &= G(1, 0)B(1, 0) + G(0, 1)B(0, 1) \end{aligned}$$

- \* We used 3.38 for the Green's function at the time  $i + j = 2$  to calculate back to the next earlier time.
- \* the Green's functions at the earlier time (here  $G(1, 0)$  at time  $i + j = 1$ ) are already known from the earlier rekursion step.
- \* It remains the arbitrage condition for the instantaneous discount factors  $B(1, 0)$  and  $B(0, 1)$ .

- Generalization to  $n$  time steps

– Separate the boundary terms in 3.40 and using 3.38:

$$\begin{aligned} B(t, t + n\delta t) &= \sum_{i=0}^n G(i, n - i) \\ &= G(0, n) + \sum_{i=1}^{n-1} G(i, n - i) + G(n, 0) \\ &= \frac{1}{2}G(0, n - 1)B(0, n - 1) \\ &\quad + \frac{1}{2} \sum_{i=1}^{n-1} G(i, n - i - 1)B(i, n - i - 1) \\ &\quad + \frac{1}{2} \sum_{i=1}^{n-1} G(i - 1, n - i)B(i - 1, n - i) \\ &\quad + \frac{1}{2}G(n - 1, 0)B(n - 1, 0) \end{aligned}$$

- use the index  $k = i - 1$  in the second sum

$$\begin{aligned}
B(t, t + n\delta t) &= \frac{1}{2}G(0, n - 1)B(0, n - 1) + \frac{1}{2}\sum_{i=1}^{n-1} G(i, n - i - 1)B(i, n - i - 1) \\
&\quad + \frac{1}{2}\sum_{k=0}^{n-2} G(k, n - k - 1)B(i - 1, n - k - 1) + \frac{1}{2}G(n - 1, 0)B(n - 1, 0)
\end{aligned}$$

- now the the boundary terms just extend the index range in the sums to 0 through  $n - 1$ , resulting in a simple recursion formula:

$$B(t, t + n\delta t) = \sum_{i=0}^{n-1} G(i, n - i - 1)B(i, n - i - 1) \quad (3.41)$$

- **This is the central equation for forward induction!**

- On the left is the market price of a zero bond.
- On the right the (to be determined!) instantaneous discount factors and Greens functions.

- The right hand side contains only nodes in the time slice *previous* to the maturity  $t + n\delta t$  of the zero bond on the left:

$$t + [i + (n - i - 1)] \delta t = t + (n - 1)\delta t$$

- The *Green's function* for this earlier time point has already been calculated in the previous iteration step.
- Thus Equation 3.41 is the arbitrage condition for the instantaneous *discount factors* at the nodes at time point  $t + (n - 1)\delta t$ .
- Having determined the discount factors from this arbitrage condition, one can then use the recursion 3.38 to determine the *Green's function* at the time point  $t + n\delta t$ .
- These are then used again in the arbitrage condition 3.41 to calculate the next discount factors for the time step  $t + n\delta t$ .
- With these then again the next *Green's function* values for  $t + (n + 1)\delta t$  are calculated with 3.38; and so on.
- In this way, an arbitrage-free interest rate tree is constructed using *forward induction*.

- But: 3.41 is only *one* arbitrage condition for  $n$  unknown discount factors.
  - \* Exact reproduction of the spot term structure is already attained singly from the fact the instantaneous discount factors and the Green's function at each node satisfy Equations 3.41 and 3.38.
  - \* This does not fix the *numerical* values of all instantaneous discount factors - or of  $r(i, j)$ .
  - \* Additional information is necessary.
    - market information about volatility
    - concrete specification of a stochastic process for the short rate, i.e. concrete specification of a term structure model.
- We will first introduce a few more concepts which hold in general, i.e. for every interest rate tree, independent of the specification of a certain stochastic process.

## 3.7 Market Rates versus Instantaneous Rates

- Underlying in the tree is the *instantaneous* short rate.

- Pricing instruments with Green's function as in 3.39 is only possible
  - \* if payoff is a function of the *instantaneous* short rate, or
  - \* if payoff is independent of the short rate (like a zero bond with  $f(i, j) = 1$  for  $t + (i + j)\Delta t = T$  and  $f(i, j) = 0$  otherwise)
- For most *traded* interest rate instruments the underlying is *not* the *instantaneous* short rate but, for example, the 3- or 6-month LIBOR.
- In contrast to e.g. stock options, here the stochastic process of the model does *not* describe the underlying of the instrument being priced!
  - No problem for bonds, floaters FRAs, swaps, etc. since they can be interpreted as a combination of zero bonds and thus can be priced *exactly* using 3.41.
  - This decomposition into zero bonds not possible for options.
- Before the payoff at a node  $(m, n)$  can be calculated, the value of the *underlying* on this nodes must first be determined!

- How to calculate an interest rate over a *finite* time period from the stochastic process (the tree) of the *instantaneous* short rate?
- As soon as this question is answered, the rest is easy:
  - determine the instrument's payoff at all relevant nodes
  - discount the payoff back to node  $(0, 0)$  using the Green's function as in 3.39 to directly get the price of the instrument at time  $t$
- Idea: As with the node  $(0, 0)$ , the value of all zero bonds (and thus all interest rates over arbitrary periods) would be known for an *arbitrary* node  $(m, n)$  if the "Green's functions" were known for *this* nodes.

### 3.7.1 Arrow-Debreu Prices

- *Arrow-Debreu prices* (short *ADPs*) are generalized Green's functions whose reference point is an arbitrary node  $(m, n)$  rather than the origin node  $(0, 0)$ .
- The ADP  $G_{m,n}(i, j)$  is defined as

- the value at node  $(m, n)$  of an instrument
- paying one monetary unit at node  $(i, j)$ .
- The ADPs at the node  $(m = 0, n = 0)$  are of course just the Green's function

$$G(i, j) = G_{0,0}(i, j)$$

- Geometry of the tree enforces:

$$\begin{aligned} G_{m,n}(i, j) &= 0 \quad \forall m > i \\ G_{m,n}(i, j) &= 0 \quad \forall n > j \\ G_{i,j}(i, j) &= 1 \quad \forall i, j \end{aligned} \tag{3.42}$$

- because a monetary unit at node  $(i, j)$  can generate non-zero prices only at nodes  $(m, n)$  if  $(i, j)$  is attainable when starting from the point  $(m, n)$ :
  - \* At  $(m, n)$  already  $m$  up moves have occurred.
    - These cannot be undone, even if the following steps consist only of down moves.

- Therefore only nodes  $(i, j)$  with  $i \geq m$  are attainable from  $(m, n)$ .
- \* Likewise,  $n$  down moves have already occurred at node  $(m, n)$  which can not be undone.
- Therefore only nodes  $(i, j)$  with  $j \geq n$  are attainable from  $(m, n)$ .
- the last property in 3.42 is trivial.

- A further fundamental property follows from the Backward Induction 3.33.

- Set either  $V(i+1, j) = 1$  and  $V(i, j+1) = 0$  or vice versa to get the so-called *instantaneous* Arrow-Debreu prices, i.e. the ADPs over one time step  $\delta t$

$$G_{i,j}(i+1, j) = \frac{1}{2}B(i, j) = G_{i,j}(i, j+1) \quad (3.43)$$

- We now require a generalization of the recursion relation 3.38 to determine all subsequent ADPs.

- Consider one monetary unit at the node  $(i, j)$ .
- Non-zero ADPs are generated at exactly two nodes in the previous time slice, namely at  $(i, j-1)$  and  $(i-1, j)$ .

- The *sum* of the ADPs of *both* of these at a still earlier node  $(m, n)$  must be equal to the ADP at node  $(m, n)$  of the whole, original monetary unit at the node  $(i, j)$ , see Figure 3.2.
- Therefore ADPs obey the following recursion:

$$G_{m,n}(i, j) = G_{m,n}(i, j - 1)G_{i,j-1}(i, j) + G_{m,n}(i - 1, j)G_{i-1,j}(i, j)$$

- Substituting the instantaneous ADPs from 3.43 finally yields

$$G_{m,n}(i, j) = G_{m,n}(i, j - 1)\frac{1}{2}B(i, j - 1) + G_{m,n}(i - 1, j)\frac{1}{2}B(i - 1, j) \quad (3.44)$$

for  $i \geq m, j \geq n$

- Together with the "geometric" properties 3.42, this recursion uniquely determines all ADPs.
- For example, at the "border of attainability", i.e. for  $m = i$  or  $n = j$ , we get

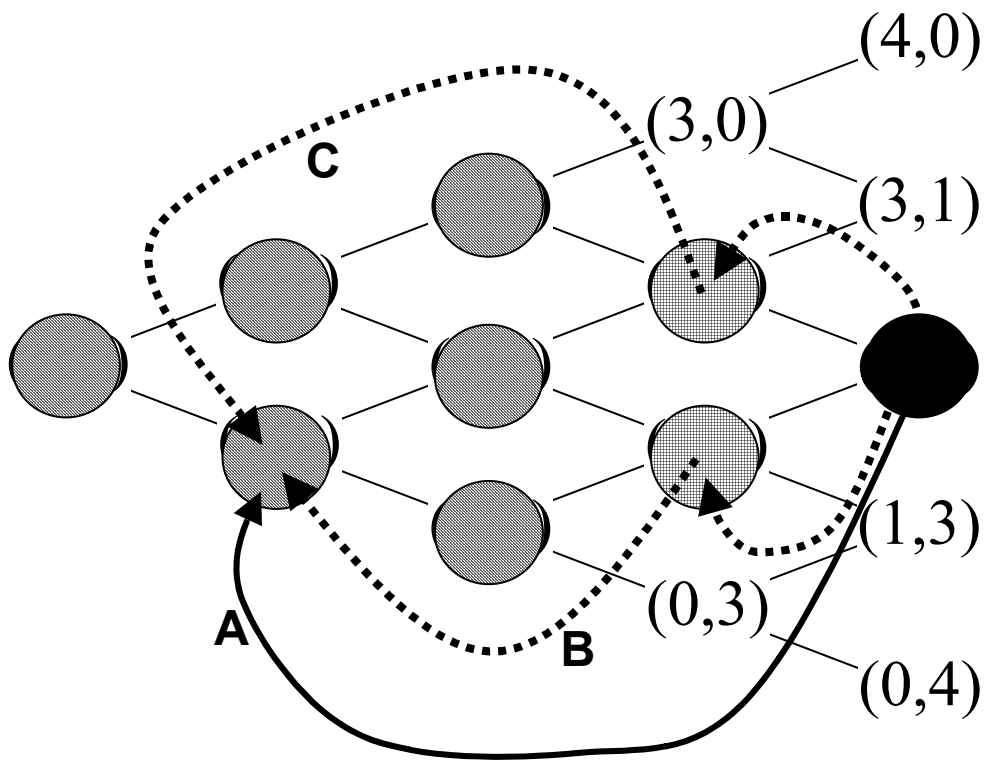


Figure 3.2: A monetary unit (black dot) at node (2,2) generates ADPs at the nodes (2,1) and (1,2) and also at all the earlier 'striped' nodes. The monetary unit must have the same influence (shown as the line **A**) on a striped node, e.g. on node (0,1), as both ADPs it has induced at the nodes (2,1) and (1,2) together (shown as the lines **B** und **C**).

using 3.42 (useful for coding an implementation):

$$\begin{aligned}
 G_{i,n}(i, j) &= G_{i,n}(i, j-1) \frac{1}{2} B(i, j-1) = \frac{1}{2^{j-n}} \prod_{k=1}^{j-n} B(i, j-k) \\
 G_{m,j}(i, j) &= G_{m,j}(i-1, j) \frac{1}{2} B(i-1, j) = \frac{1}{2^{i-m}} \prod_{k=1}^{i-m} B(i-k, j) \quad (3.45)
 \end{aligned}$$

\* In particular, this holds at the boundary of the tree:

- the first equation with  $i = 0$  (no up move) represents the "lower" boundary,
- the second with  $j = 0$  (no down move) the "upper" boundary of the tree.

- The value of a financial instrument with payoff  $f(i, j)$  at an *arbitrary* node  $(m, n)$  is now (analogous to 3.39) simply

$$V(m, n) = \sum_{(i,j)} G_{m,n}(i, j) f(i, j) \quad (3.46)$$

- A zero bond at the node  $(m, n)$  with time-to-maturity  $\tau$  at this node has a payoff function equal to zero everywhere except for  $f(i, j) = 1$  at nodes fulfilling

$$i + j = m + n + \tau/\delta t$$

- Because of the property 3.42 the indices fulfill two more conditions:

$$i \geq m$$

$$j \geq n$$

- The value  $B_\tau(m, n)$  in 3.46 of the zero bond at the node  $(m, n)$  with a time-to-maturity of  $\tau$  is thus explicitly

$$B_\tau(m, n) = \sum_{i=m}^{m+\tau/\delta t} G_{m,n}(i, m + n + \tau/\delta t - i) \quad (3.47)$$

- The lower limit of summation is  $i \geq m$ , the upper limit follows from  $m + n + \tau/\delta t - i = j \geq n$  which can be re-written as  $i \leq m + \tau/\delta t$ .

- Thus, at each node, a complete term structure (i.e. future interest rates for arbitrary times to maturity) can be constructed from ADPs.

– For example in continuous compounding:

$$r_\tau(m, n) = -\frac{1}{\tau} \ln B_\tau(m, n) \quad (3.48)$$

– Therefore any underlying deriveable from the term structure (for example, 3-month LIBOR, a swap rate, etc.) can be constructed

### 3.7.2 Pricing Caplets by means of the Arrow-Debreu Prices

- As an example we will now explicitly express the price of a caplet with a principle of  $N$  and strike price  $K$  on a 3-month rate with exercise at time  $T$  and payment date  $T' = T + \tau$  (with  $\tau = 3$  months) in terms of ADPs.

– *No* stochastic process for the short rate needs to be specified to do this.

– What we will show holds for *every arbitrary* arbitrage-free short rate model!

- The interest rates determining the payoff of a caplet are by market convention always fixed using *simple compounding* over the single caplet period (in this case 3 months).
- Payoff is determined at nodes  $(m, n)$  with  $t+(m+n)\delta t = T$ , i.e.  $n = (T-t)/\delta t - m$  is thus:
- The 3-month rate in simple compounding at these nodes is:

$$\begin{aligned}
 r_\tau(m, n) &= \frac{1}{\tau} [B_\tau(m, n)^{-1} - 1] & (3.49) \\
 &= \left( b\delta t \sum_{i=m}^{m+b} G_{m, a-m}(i, a + b - i) \right)^{-1} - \frac{1}{b\delta t} \\
 &\text{for all } 0 \leq m \leq a
 \end{aligned}$$

– with

$$a := (T - t)/\delta t \quad , \quad b := \tau/\delta t$$

- The payoff profile, i.e. the values of the caplet at the exercise date  $T$ , i.e. at the nodes  $(m, a - m)$ , for  $0 \leq m \leq a$  is

$$f(m, a - m) = \tau N B_\tau(m, a - m) \max \{r_\tau(m, a - m) - K, 0\}$$

- Both the discount factor  $B_\tau$  over the caplet period as well as the underlying  $r_\tau$  can be expressed in terms ADPs:

$$f(m, a - m) = N \sum_{i=m}^{m+b} G_{m,a-m}(i, a + b - i) \times \max \left\{ \left( \sum_{i=m}^{m+b} G_{m,a-m}(i, a + b - i) \right)^{-1} - 1 - \tau K, 0 \right\}$$

- Equation 3.39 now directly yields the value of this payoff, i.e. caplet value, at

time  $t$

$$\begin{aligned} c^{\text{cap}}(T, T + \tau, K | t) &= \sum_{m=0}^a G(m, a - m) f(m, a - m) \\ &= N \sum_{m=0}^a G_{0,0}(m, a - m) \sum_{i=m}^{m+b} G_{m,a-m}(i, a + b - i) \quad (3.50) \\ &\times \max \left\{ \left( \sum_{i=m}^{m+b} G_{m,a-m}(i, a + b - i) \right)^{-1} - 1 - \tau K, 0 \right\} \end{aligned}$$

### Practical Implementation of the Arrow-Debreu Prices

- ADPs have 4 indices.
  - Numerical implementation of a four-dimensional field leads to computer memory problems for finer trees as well as performance problems in the computation of all the ADPs.
- In most cases it is unnecessary to compute all ADPs.

- For caplets, for example, only ADPs of the form  $G_{m,a-m}(i, a + b - i)$  with the fixed parameters  $a$  and  $b$  are needed.
- Thus, only *two* indices are free, namely  $m$  and  $i$ .
- Sufficient to construct the two-dimensional field

$$\tilde{G}_{m,i} := G_{m,a-m}(i, a + b - i)$$

- The numerical effort is now the same as that needed for the Green's function.

### 3.8 Explicit Specification of Short Rate Models

- Up to this point no specific term structure has been used.
- As yet no explicit numerical value can be calculated since the arbitrage condition 3.41 is not sufficient to determine all instantaneous discount factors.
- From now on, we suppose that the term structure model is of the general form

$$dS(t) = a(S, t) dt + b(S, t) dW \tag{3.51}$$

- with previsible processes  $a$  and  $b$
- The stochastic variable  $S$  is either the logarithm of the short rate or the short rate itself.

- In so-called *normal models* the short rate itself follows a process

$$dr(t) = a(r, t) dt + b(r, t) dW \quad (3.52)$$

- parameter  $b$  represents an *absolute* volatility
- If  $a$  and  $b$  independent of  $r$  : *Ho-Lee Model*.

\* Advantages:

- Easy to implement (for example, as a tree).
- For constant  $b$  is even analytically solvable.

\* Drawbacks:

- negative interest rates possible

- So-called *lognormal models* do not specify the short rate itself, rather its logarithm:

$$d \ln r(t) = a(r, t) dt + b(r, t) dW \quad (3.53)$$

- parameter  $b$  represents a *relative* volatility
  - If volatility  $b$  is independent of  $r$ : *Black–Derman–Toy model*, although, strictly speaking, the original Black-Derman-Toy model employs a very special form of drift.
  - Drawbacks:
    - \* significantly more difficult to implement
    - \* can not be solved analytically.
  - Advantages:
    - \* negative interest rates impossible
- Lognormal models can be transformed into models for  $r$  via Ito’s Lemma<sup>8</sup>:

$$dr(t) = \left[ r(t)a(r, t) + \frac{1}{2}b(r, t)^2r(t) \right] dt + r(t)b(r, t)dW \quad (3.54)$$

---

<sup>8</sup>For this, we need to consider the following:  $\ln r(t)$  in Equation 3.53 corresponds to the stochastic variable  $S$ . For the function  $f$  take  $f(S) = e^S$  (since this is exactly  $r$ ). The partial derivatives needed in Ito’s formula are then simply  $\partial f/\partial t = 0$  and  $\partial f/\partial S = \partial^2 f/\partial S^2 = f = r$ .

- The numerical evaluation of lognormal models is a lot easier to implement when written in this form.

### 3.8.1 The Effect of the Volatility

- According to the Girsanov Theorem, the drift  $a(r, t)$  is uniquely determined by the probability measure used.
  - We fixed this measure "by hand" when we required  $p = 1/2$  in Equation 3.27.
  - Thus we have implicitly fixed the drift as well. The drift can thus no longer be an "input".
- Only the volatility remains through which market information (in addition to the bond prices) may enter.
  - We will show that volatility input at time step  $n$  gives exactly  $n - 1$  conditions for the  $n$  instantaneous discount factors.

- Together with the arbitrage condition 3.41 for the bond prices we finally have exactly as many conditions as unknowns.
  - The interest rate tree can be uniquely constructed.
- In general in a binomial tree, the variance of any variable  $x$  at node  $(i, j)$  is caused by its two possible values in the next time step,  $x_u$  or  $x_d$ .
    - Expectation and variance are therefore<sup>9</sup>

$$\begin{aligned} \mathbb{E}[x] &= px_u + (1-p)x_d \\ \text{Var}[x] &= p(1-p)(x_u - x_d)^2 \end{aligned} \tag{3.55}$$

---

<sup>9</sup>Substituting the expectation into the definition of the variance expressed as the expectation of the square of the deviation gives

$$\begin{aligned} \text{Var}[x] &= \mathbb{E}[(x - \mathbb{E}[x])^2] \\ &= p(x_u - \mathbb{E}[x])^2 + (1-p)(x_d - \mathbb{E}[x])^2 \\ &= p(x_u - px_u - (1-p)x_d)^2 + (1-p)(x_d - px_u - (1-p)x_d)^2 \end{aligned}$$

Multiplying out and collecting the terms yields the desired expression.

– For  $p = 1/2$ , the variance is simply

$$\text{Var}[x] = (x_u - x_d)^2/4$$

- On the other hand, in our models 3.52 or 3.53, the variance of  $x$  (with  $x = r$  resp.  $x = \ln r$ ) over an time interval  $\delta t$  is  $b(r, t)^2 \delta t$ .

### 3.8.2 Normal Models

- For models of the form 3.52 the variance of the short rate at the node  $(i, j)$  must satisfy:

$$\begin{aligned} b(i, j)\sqrt{\delta t} &= \sqrt{\text{Var}[r(i, j)]} = \frac{1}{2} [r(i+1, j) - r(i, j+1)] \\ &\implies \\ r(i+1, j) &= r(i, j+1) + 2b(i, j)\sqrt{\delta t} \end{aligned} \tag{3.56}$$

- This gives a recursion formula (after the substitution  $i + 1 \rightarrow i$ ) for the instantaneous discount factors in the time slice  $n = i + j$

$$\begin{aligned}
 B(i, j) &= \exp \{-r(i, j)\delta t\} \\
 &= \exp \left\{ - \left[ r(i-1, j+1) + 2b(i-1, j)\sqrt{\delta t} \right] \delta t \right\} \\
 &= \exp \left\{ -2b(i-1, j)\sqrt{\delta t}\delta t \right\} \exp \{-r(i-1, j+1)\delta t\} \\
 &= \exp \left\{ -2b(i-1, j)\sqrt{\delta t}\delta t \right\} B(i-1, j+1)
 \end{aligned}$$

- or

$$B(i, j) = \alpha(i-1, j)B(i-1, j+1) \quad \text{with} \quad \alpha(i, j) = \exp \{-2b(i, j)\delta t^{3/2}\} \quad (3.57)$$

- Recursive substitution expresses each instantaneous discount factor in this time

slice in terms of the discount factor at the "lowest" node  $(0, i + j)$ :

$$\begin{aligned}
 B(i, j) &= \alpha(i - 1, j)B(i - 1, j + 1) \\
 &= \alpha(i - 1, j)\alpha(i - 2, j + 1)B(i - 2, j + 2) \\
 &= \alpha(i - 1, j)\alpha(i - 2, j + 1)\alpha(i - 3, j + 2)B(i - 3, j + 3) \\
 &= \dots \\
 &= B(0, j + i) \prod_{k=1}^i \alpha(i - k, j + k - 1)
 \end{aligned}$$

- Use this  $B(i, j)$  with  $j = n - i - 1$  in 3.41 to get:

$$B(t, t + n\delta t) = B(0, n - 1) \sum_{i=0}^{n-1} G(i, n - i - 1) \prod_{k=1}^i \alpha(i - k, n - i + k - 2)$$

- After a simple change of index  $n \rightarrow (n + 1)$ , we finally get the arbitrage condition

for the discount factor  $B(0, n)$  at the lowest node in time slice  $n$ :

$$B(t, t + (n + 1)\delta t) = B(0, n) \sum_{i=0}^n G(i, n - i) \prod_{k=1}^i \alpha(i - k, n - i + k - 1) \quad (3.58)$$

- **This is the central equation to solve when implementing a normal model.**
- It can easily be solved analytically for the only unknown left:  $B(0, n)$ .
- From  $B(0, n)$ , all further discount factors in this time slice follow from 3.57.
- The information flow is shown in Figure 3.3.
  - \* The left hand side is the *market* price of a zero bond with maturity at the time slice  $(n + 1)$ .
  - \* This must be reproduced with Green's function at time slice  $n$ .
  - \* The  $\alpha$ 's (volatility information) all belong to time slice  $(n - 1)$ .
    - Only volatility information from the *previous* time step is required!
    - This is the *previsibility* which we always require for the coefficients  $a$  and  $b$ .



- From the discount factors the short rates results immediately.
- For continuous compounding with 3.57

$$\begin{aligned}
r(i, j) &= -\frac{1}{\delta t} \ln B(i, j) = -\frac{\ln B(i-1, j+1)}{\delta t} - \frac{\ln \alpha(i-1, j)}{\delta t} \\
&= r(i-1, j+1) + 2b(i-1, j)\sqrt{\delta t} \\
&= \dots \\
&= r(0, j+i) + 2\sqrt{\delta t} \sum_{k=1}^i b(i-k, j+k-1)
\end{aligned}$$

### Normal Models with an $r$ -independent volatility structure

- If volatility depends only on time the  $b$  values in one time slice are all identical.

$$\begin{aligned}
b(i, j) &= b(0, i+j) = b(0, n) \equiv \sigma(t+n\delta t) \\
\alpha(i, j) &= \alpha(0, i+j) = \alpha(0, n) \quad \forall i, j \text{ with } i+j=n
\end{aligned} \tag{3.59}$$

- 3.57 reduces to

$$\begin{aligned} B(i, j) &= \alpha(0, n-1)B(i-1, j+1) \\ &= \alpha(0, n-1)^i B(0, n) \quad \text{with } i+j=n \end{aligned}$$

- The short rate in 3.56 at the time slice  $n$  changes from node to node by a constant term  $2\sigma\sqrt{\delta t}$

$$\begin{aligned} r(i, j) &= r(i-1, j+1) + 2\sigma(t + (n-1)\delta t)\sqrt{\delta t} \\ &= r(0, n) + 2i\sigma(t + (n-1)\delta t)\sqrt{\delta t} \end{aligned} \tag{3.60}$$

- The arbitrage condition for this discount factor at the lowest node reduces to

$$B(t, t + (n+1)\delta t) = B(0, n) \sum_{i=0}^n G(i, n-i)\alpha(0, n-1)^i$$

### 3.8.3 Lognormal Models

- In models of the form 3.53,  $b(i, j)^2 \delta t$  is the variance of the *logarithm* of the short rate:

$$\begin{aligned} b(i, j) \sqrt{\delta t} &= \sqrt{\text{Var} [\ln r(i, j)]} = \frac{1}{2} [\ln r(i + 1, j) - \ln r(i, j + 1)] = \frac{1}{2} \ln \left( \frac{r(i + 1, j)}{r(i, j + 1)} \right) \\ &\implies \\ r(i + 1, j) &= r(i, j + 1) \exp \left\{ 2b(i, j) \sqrt{\delta t} \right\} \end{aligned} \tag{3.61}$$

- Recursion (after performing the substitution  $i + 1 \rightarrow i$ ) for the discount factors

in the time slice  $n = i + j$ <sup>10</sup>

$$\begin{aligned} B(i, j) &= \exp \{-r(i, j)\delta t\} \\ &= \exp \left\{ -r(i-1, j+1)e^{2b(i-1, j)\sqrt{\delta t}}\delta t \right\} \\ &= [\exp \{-r(i-1, j+1)\delta t\}]^{e^{2b(i-1, j)\sqrt{\delta t}}} \\ &= B(i-1, j+1)e^{2b(i-1, j)\sqrt{\delta t}} \end{aligned}$$

• or

$$B(i, j) = B(i-1, j+1)^{\alpha(i-1, j)} \quad \text{with} \quad \alpha(i, j) = \exp \{+2b(i, j)\delta t^{1/2}\} \quad (3.62)$$

- Similar to the corresponding 3.57, but the volatility information enters now as an *exponent* rather than a factor.
- Furthermore: a sign change in the expression for  $\alpha$ .

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<sup>10</sup>We have used that, for the exponential function,  $\exp\{ax\} = (\exp\{x\})^a$  holds with  $a = e^{2b(i-1, j)\sqrt{\delta t}}$ .

- Trick: take logarithms to get a similar structure as Equation 3.57:

$$\ln B(i, j) = \alpha(i-1, j) \ln B(i-1, j+1)$$

– Now carry out this recursion:

$$\begin{aligned} \ln B(i, j) &= \alpha(i-1, j) \ln B(i-1, j+1) \\ &= \alpha(i-1, j) \alpha(i-2, j+1) \ln B(i-2, j+2) \\ &= \alpha(i-1, j) \alpha(i-2, j+1) \alpha(i-3, j+2) \ln B(i-3, j+3) \\ &= \dots \\ &= [\ln B(0, j+i)] \prod_{k=1}^i \alpha(i-k, j+k-1) \end{aligned}$$

- Allowing the instantaneous discount factors in the time slice  $n$  to be written as a function of the "lowest" node  $(0, i+j)$ :

$$\begin{aligned} B(i, j) &= \exp \left\{ [\ln B(0, j+i)] \prod_{k=1}^i \alpha(i-k, j+k-1) \right\} \\ &= B(0, j+i) \prod_{k=1}^i \alpha(i-k, j+k-1) \end{aligned}$$

- Use this  $B(i, j)$  with  $j = n - i - 1$  in 3.41, to get (after the index transformation  $n \rightarrow n + 1$ ) the arbitrage condition for the discount factor  $B(0, n)$  at the lowest node in the time slice  $n$ .

$$B(t, t + (n + 1)\delta t) = \sum_{i=0}^n G(i, n - i) \exp \left\{ [\ln B(0, n)] \prod_{k=1}^i \alpha(i - k, n - i + k - 1) \right\} \quad (3.63)$$

$$= \sum_{i=0}^n G(i, n - i) B(0, n)^{\prod_{k=1}^i \alpha(i - k, n - i + k - 1)}$$

- **This is the central equation to solve when implementing a lognormal model.**
- It can only be solved numerically for the only unknown left:  $B(0, n)$ .
- Use the well-known *Newton-Raphson method*<sup>11</sup>
- Again the discount factors on the time slice  $n$  are calculated from the price of the zero bond maturing on the *following* time slice ( $n + 1$ ) and from the

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<sup>11</sup>To solve a non-linear equation of the form  $f(x) = 0$ , the *Newton-Raphson method* uses the following iteration to find the points where the function  $f$  equals zero: Having an estimate  $x_i$  for a zero of

volatility information from the immediately *preceding* time slice  $(n - 1)$ , see Figure 3.3.

- The short rates now follow immediately from 3.62. For instance in continuous

---

$f$ , a better estimate is obtained from the formula

$$x_{i+1} = x_i - f(x_i) \left( \frac{\partial f}{\partial x} \Big|_{x=x_i} \right)^{-1}$$

We usually start the procedure with a rough estimate,  $x_0$ , and iterate until the difference between  $x_{i+1}$  and  $x_i$  is sufficiently small for our purposes. The iteration sequence converges if

$$\left| f \frac{\partial^2 f / \partial^2 x}{(\partial f / \partial x)^2} \right| < 1$$

holds in a neighborhood of the zero. This can be always assumed to be the case in our applications.

compounding:

$$\begin{aligned}
 r(i, j) &= -\frac{1}{\delta t} \ln B(i, j) = -\alpha(i-1, j) \frac{\ln B(i-1, j+1)}{\delta t} \\
 &= r(i-1, j+1) \alpha(i-1, j) \\
 &= \dots \\
 &= r(0, j+i) \prod_{k=1}^i \alpha(i-k, j+k-1)
 \end{aligned}$$

### Exact Reproduction of the Term Structure with the Lognormal Model

- Alternatively, proceed from the process 3.54 for the short rate *itself* obtained from Ito's formula
- All equations are then *exactly* as for the *normal* model (in particular 3.58)
- The *only* difference is capsulated into  $\alpha$ , which is now

$$\alpha(i, j) = \exp - 2b(i, j)r(i, j)\delta t^{3/2} \tag{3.64}$$

- The short rates  $r(i, j)$  needed for this equation were calculated in the *previous* iteration step, so all terms are known.
- Thus – although in the context of the lognormal model – we can be solve *analytically* for the lowest bond!

### Lognormal Models with $r$ -independent Volatility Structure

- For only time (but not interest rate) dependent volatility structure, Equation 3.57 reduces to

$$\begin{aligned} B(i, j) &= B(i - 1, j + 1)^{\alpha(0, n-1)} \\ &= B(0, n)^{\alpha(0, n-1)^i} \quad \text{with } i + j = n \end{aligned}$$

– with  $\alpha(i, j) = \exp \left\{ 2\sigma(t + n\delta t)\sqrt{\delta t} \right\}$ .

- The short rate is simply to be multiplied by a constant factor  $\alpha(0, n - 1)$  when

moving from one node to the next in one time slice

$$\begin{aligned}r(i, j) &= r(i - 1, j + 1)\alpha(0, n - 1) \\ &= r(0, n)\alpha(0, n - 1)^i\end{aligned}\tag{3.65}$$

- The arbitrage condition for the lowest bond reduces to

$$B(t, t + (n + 1)\delta t) = \sum_{i=0}^n G(i, n - i)B(0, n)^{\alpha(0, n-1)^i}$$

### 3.9 Explicit Step-by-Step Construction

- Input
  - the term structure, in particular the market prices of zero bonds

$$B(t, t + i\delta t) \quad \text{for } i = 1 \dots n$$

– the volatilities

$$b(r, t + i\delta t) \quad \text{for } i = 0 \dots n - 1$$

\* calculate  $\alpha$  via 3.57 or 3.62

• Time step  $i = 0$

– 3.42 for the Green's function at the node  $(0, 0)$

$$G(0, 0) = 1$$

– 3.41 for the discount factor at  $(0, 0)$ :

$$B(t, t + 1\delta t) = G(0, 0)B(0, 0) = B(0, 0)$$

• Time step  $i = 1$

- substitute the just computed discount factors and Green's function into the recursion 3.38 to get the Green's function at the current time step:

$$G(1, 0) = \frac{1}{2}G(0, 0)B(0, 0)$$
$$G(0, 1) = \frac{1}{2}G(0, 0)B(0, 0)$$

- substitute
  - \* these Green's function values,
  - \* the volatility input from the previous time step,
  - \* the market price of the zero bond maturing at the next time step
- into the arbitrage condition 3.58 (for normal models) to get the discount factor at the lowest node:

$$B(0, 1) = \frac{B(t, t + 2\delta t)}{G(0, 1) + G(1, 0)\alpha(0, 0)}$$

- The recursion 3.57 (for normal models) now immediately yields the other discount factor for this time step:

$$B(1, 0) = \alpha(0, 0)B(0, 1)$$

- for lognormal models the arbitrage condition 3.63 for the lowest bond is:

$$B(t, t + 2\delta t) = G(0, 1)B(0, 1) + G(1, 0)B(0, 1)^{\alpha(0, 0)}$$

\* solve numerically (with Newton Raphson, for example) for  $B(0, 1)$ .

- for lognormal models the recursion 3.62 now immediately yields the other discount factor for this time step:

$$B(1, 0) = B(0, 1)^{\alpha(0, 0)}$$

- Time step  $i = 2$

- substitute the just computed discount factors and Green's function into the

recursion 3.38 to get the Green's function at the current time step:

$$G(1, 1) = \frac{1}{2}G(1, 0)B(1, 0) + \frac{1}{2}G(0, 1)G(0, 1)$$

$$G(2, 0) = \frac{1}{2}G(1, 0)B(1, 0)$$

$$G(0, 2) = \frac{1}{2}G(0, 1)B(0, 1)$$

– substitute

- \* these Green's function values,
- \* the volatility input from the previous time step,
- \* the market price of the zero bond maturing at the next time step

– into the arbitrage condition 3.58 (for normal models) to get the discount

factor at the lowest node:

$$\begin{aligned}
 B(t, t + 3\delta t) &= B(0, 2) \sum_{i=0}^2 G(i, 2 - i) \prod_{k=1}^i \alpha(i - k, 1 - i + k) \\
 &= B(0, 2)G(0, 2) + B(0, 2)G(1, 1)\alpha(0, 1) \\
 &\quad + B(0, 2)G(2, 0)\alpha(1, 0)\alpha(0, 1) \\
 &\implies
 \end{aligned}$$

$$B(0, 2) = \frac{B(t, t + 3\delta t)}{G(0, 2) + G(1, 1)\alpha(0, 1) + G(2, 0)\alpha(1, 0)\alpha(0, 1)}$$

- The recursion 3.57 (for normal models) now immediately yields the other discount factors for this time step:

$$B(1, 1) = \alpha(0, 1)B(0, 2)$$

$$B(2, 0) = \alpha(1, 0)B(1, 1)$$

- for lognormal models the arbitrage condition 3.63 for the lowest bond is:

$$B(t, t + 3\delta t) = G(0, 2)B(0, 2) + G(1, 1)B(0, 2)^{\alpha(0, 1)} + G(2, 0)B(0, 2)^{\alpha(1, 0)\alpha(0, 1)}$$

- \* solve numerically (with Newton Raphson, for example) for  $B(0, 2)$ .
- for lognormal models the recursion 3.62 now immediately yields the other discount factors for this time step::

$$B(1, 1) = B(0, 2)^{\alpha(0,1)}$$

$$B(2, 0) = B(1, 1)^{\alpha(1,0)}$$

- repeat these steps until the entire tree has been constructed up to maturity  $T + \tau$  (Maturity of the derivative to be priced plus the lifetime of the underlying).
- determine all required Arrow-Debreu prices with Equations 3.42 and 3.44
- finally price the derivative as demonstrated for caplets in in 3.50.

### 3.9.1 Absolute and Relative Volatilities

- Volatility input for lognormal models is required to be a *relative* volatility
  - for example, 14% of its current value.

- For normal models, an *absolute* volatility is required.
  - for example, 0.75 percentage *points*<sup>12</sup>
  - a *relative* volatility input must first be multiplied with an interest rate to make it absolute. Which one?
    - \* The instantaneous short rate? At which node?
    - \* The underlying (for example, the 3-month rate)? At which time? As of today?
    - \* Use the *forward rate of the underlying at the maturity of the option* as the multiplicative factor<sup>13</sup>!
      - This forward rate is used as the underlying in the Black-76 model.
      - Therefore the Black-76 volatility quoted in the market belongs to this forward rate.

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<sup>12</sup>At a current interest rate of 6% this would correspond to a relative volatility of 14%.

<sup>13</sup>Observe the *compounding method* being used to obtain the forward rate! For instance for linear compounding (as used over FRA, Swap or IRG periods) use  $r = (B^{-1} - 1) / \tau$  to get the correct forward rate from the forward discount factors.

- A lognormal model in the form 3.54 (from Ito's Lemma) has relative volatilities at any node  $(i, j)$  always appears in a product together with the instantaneous  $r(i, j)$ .
  - Otherwise, everything is *exactly* the same as in a normal model.
  - This product of relative volatility and short rate can be interpreted as an *absolute* volatility.
  - Therefore: In the *lognormal* model, the conversion factor between the relative and the absolute volatility is the short rate  $r(i, j)$  at each node  $(i, j)$ ;
    - \* in contrast to the normal model, for which this conversion factor is a constant (for example, the forward rate of the underlying).
- The difference in the implementation of a normal and lognormal model is only one single line of code (defining the conversion factor between relative and absolute vols) when the lognormal model is constructed according to 3.54!

### 3.9.2 Calibration of the Volatilities

- The volatilities  $b(r, t + i\delta t)$  are usually not known for the model under consideration.
- Only the prices of options traded on the market (caps, floors, swaptions, etc.) can be observed directly.
- The tree must then be constructed, leaving the volatilities unspecified as free parameters.
- These are then adjusted (with Newton Raphson, for example) until the observed market prices of options are reproduced by the model.
- Fitting the parameters  $b(r, t + i\delta t)$  to the market prices is referred to as the *calibration* of the volatility.
- There are many procedures how such a calibration can be done.
- One possibility: reproduce the market prices of the options stepwise through the tree, beginning with the shortest option lifetimes and proceeding through to the longest.

- Assume we have market prices of a of 3-month-caplets covering the time span under consideration without overlap, i.e.

$$T_{k+1} = T_k + \tau$$

- \* with  $T_k$  the maturity of the  $k$ -th caplet and  $\tau$  the lifetime of the underlying rate (i.e. 3 months).

- Start the calibration by assuming the volatility to be constant in the tree from  $t$  and  $T_1 + \tau$

$$b(r, t + i\delta t) = b(T_1) \quad \text{for all } r \text{ and all } i\delta t \text{ with } t \leq t + i\delta t \leq T_1 + \tau$$

- \* Adjust this constant volatility (using Newton Raphson) until the price computed with the tree equals the market price of the first caplet.

- For the second caplet price assume that the volatility remains equal to the (just calibrated)  $b(T_1)$  from  $t$  to  $T_1 + \tau$ , and is (unknown but) constant between  $T_1 + \tau = T_2$  and  $T_2 + \tau$ :

$$b(r, t + i\delta t) = b(T_2) \quad \text{for all } r \text{ and all } i\delta t \text{ with } T_1 + \tau < t + i\delta t \leq T_2 + \tau$$

- Adjust this constant volatility (using Newton Raphson) until the price computed with the tree equals the market price of the second caplet.
- To reproduce the third caplet price again assume, that all the already calibrated volatilities remain valid,
  - \* and adjusts the volatility between  $T_2 + \tau = T_3$  and  $T_3 + \tau$  until the price of the third caplet price is reproduced.
- and so on.
- By this method, we obtain a piecewise constant function for the volatility as a function of time, independent of  $r$ .
- Alternatively, the volatilities can be (simultaneously) calibrated in using a least squares fit.
  - One then minimizes the sum of the quadratic differences between the calculated and the traded option prices by varying the volatilities.
- Such a calibration yields different volatility values  $b(r, t + i\delta t)$  for each different term structure model.

- One therefore refers to the respective volatilities by the name of the model,
  - for example Ho-Lee volatilities, Black-Derman-Toy volatilities, Hull-White volatilities, etc.
  - It is, in general, not possible to simply take the quoted Black-76 volatilities as the input values for  $b(r, t + i\delta t)$ .
- Calibration with respect to other instruments, for example swaption prices is in its implementation considerably more involved, but is based on the same principle:
  1. Calculation of all required Arrow Debreu prices from the existing tree.  
Generation of the necessary underlyings, namely the swap rate under consideration, from the Arrow Debreu prices.
  2. Calculation of the payoff profiles of the swaption at the nodes corresponding to the swaption maturity date based on the value of the underlying (swap rate) and with the help of the Arrow Debreu prices

3. Calculation of the swaption prices at the node  $(0,0)$  by discounting the payoff profile with the Green's function

Adjusting the volatility in the interest tree until the swaption price computed with the tree agrees with the price quoted on the market.

### 3.10 Monte Carlo on the Tree

- By performing a *Monte-Carlo simulation on the tree*, the trees introduced here can even be used for pricing such path dependent derivatives for which the historical progression of the interest rate influences the payoff profile.
- Such a Monte-Carlo simulation works as follows:
  - First observe that the tree needs to be generated only once and then stays in the memory of the computer.
  - The interest rate paths are then simulated by jumping randomly from node to node in the tree, always proceeding one time step further with each jump.

- Simulating the jumps from node to node with the appropriate transition probabilities (in our case here  $p = 1/2$  for up- as well as for down-moves, see Equation 3.27) of the chosen martingale measure (in our case the risk-neutral measure, see Equation 3.26) ensures that the simulated paths already have the correct probability weight needed for pricing financial instruments.
  - \* This procedure is called *importance sampling*<sup>14</sup>.
- At the end of each simulated path the payoff of the derivative resulting from the underlying having taken this path is calculated.
- This payoff is then discounted back to the current time  $t$  (i.e. to node  $(0, 0)$ ) with the short rates *along the simulated path*, since after all, we still are within the risk-neutral measure, see Equation 3.26.

---

<sup>14</sup>By doing a move only with its associated probability ensures a very effective sampling of the so-called *phase space* (the set of all possible values of the simulated variables): Phase space regions (values of the simulated variables) which have low probabilities (and therefore contribute only little to the desired averages of whatever needs to be measured by the simulation) are only visited with low probability (i.e. rarely) while phase space regions with high probabilities (which contribute a lot to the desired averages) are visited with high probability (i.e. often). Because of this feature *importance sampling* is heavily used in thousands of Monte-Carlo applications, especially in physics, meteorology and other sciences which rely on large scale simulations.

- After many (usually several thousand) paths have been simulated, the thus generated (several thousand) discounted payoff-values can then simply be averaged to yield an estimation for the risk-neutral expectation of the discounted payoff.
  - \* Here the very simple arithmetic average of the discounted payoff-values can be used without any worry about the correct probability weight of each payoff-value since the payoff-values (more precisely the paths which generated the payoff-values) have already been *simulated* with the correct (risk-neutral) probability.
- The risk-neutral expectation calculated in this way is via Equation 3.8 directly the desired derivative price.

### 3.11 The Drift in Term Structure Models

- Now: reconcile the practical computations in Section 3 with the deep concepts from Section 1.
- First we examine the relationship between the underlying instantaneous rates

and tradable instruments such as zero bonds.

- For this, take a brief excursion into the models in which the instantaneous *forward* rate plays the role of the underlying.

### 3.11.1 Heath-Jarrow-Morton Models

- One recognizes from the fundamental Equations 3.2, 3.3 and 3.5 that the instantaneous *forward* rates generate the prices  $B(t, T)$  of all zero bonds as well as the entire term structure  $R(t, T)$ .
- All three descriptions of the term structure are equivalent, namely
  - zero bond prices  $B(t, T)$ ,
  - zero bond yields  $R(t, T)$ ,
  - instantaneous forward rates  $f(t, T)$ .
- Only one of these needs be chosen to be modelled the with a stochastic process.

- Take, for example, the forward rates  $f(t, T)$  to be modelled (in the *real* world) as:

$$df(t, T) = a(t, T) dt + b(t, T) dW \quad \text{with} \quad dW = X\sqrt{dt}, X \sim N(0, 1) \quad (3.66)$$

- An entire class of term structure models, the *Heath-Jarrow-Morton Models* (*HJM models*) take this approach.
- Like all interest rates, instantaneous forward rates themselves are not tradable (see Section 3.4).
- When the underlying is not tradable, one seeks a tradable instrument  $U$  whose price  $U(S, t)$  is a *function* of the underlying. Then all of the results shown in Chapter 1 hold:
  - The Harrison-Pliska Theorem establishes the *uniqueness* (in complete markets) of the probability measure with respect to which the price of tradable financial instruments normed with an arbitrarily chosen, tradable numeraire instrument  $Y$  are martingales.

- By the Girsanov Theorem, this implies that there exists only *one single underlying drift* which may be used for pricing.
- In addition, the drift of the underlying in the real world plays absolutely no role in the world governed by the martingale measure if the numeraire  $Y$  satisfies the property 2.19 (which is always the case).
- **The existence of a function  $U(S, t)$  relating the underlying  $S$  to a tradable instrument  $U$  is the deciding factor!**
- For the instantaneous forward rate  $f(t, T)$ , such a functional relation exists, namely Equation 3.3.
  - Thus, for every complete market, there exists for each numeraire instrument, exactly *one* single drift to be used for pricing.
  - Therefore, as far as *pricing* is concerned, the HJM model is uniquely determined by the volatility  $b(t, T)$  in Equation 3.66.
  - Indeed, the forward rate process in the risk-neutral measure corresponding to the real world process in Equation 3.66 has the following appearance

[15]:

$$df(t, T) = \left[ b(t, T) \int_t^T b(t, s) ds \right] dt + b(t, T) d\widetilde{W}$$

- \* Here  $d\widetilde{W}$  denotes the standard Brownian motion with respect to the martingale measure.
- \* The coefficient of  $dt$  appearing in square brackets is the drift with respect to the martingale measure.
- \* This shows *explicitly* that for HJM models, the entire model (including the drift) is uniquely specified through the volatility  $b(t, T)$ .
- \* The drift to be used in the valuation is *unique*, in complete agreement with the general statements made in Chapter 1.

### 3.11.2 Short Rate Models

- For the instantaneous *spot* rate as the underlying a one-to-one mapping between spot rates and the zero bond prices does *not* exist.
  - The instantaneous spot rates are not sufficient to generate the term struc-

ture!

- The spot rate  $r(t)$  is a function of a *single* time variable  $t$  but
  - the processes  $B(t, T)$ ,  $R(t, T)$  (and  $f(t, T)$  as well!) which are functions of *two* time variables.
  - Taking the limit  $dt \rightarrow 0$  in the definition 3.1 results in the loss of the second argument (and thus in the loss of the corresponding information).
  - This can also be seen explicitly in Equation 3.6.
- For this reason, there is *no* analog to 3.3 relating the instantaneous spot rates directly to the zero bond prices.
    - The best one can do is to determine the bond prices from the *expectations* of the short rates (see, for example Equation 3.9), but not directly as a *function* of the short rates.
    - A *direct functional relationship* between the underlying (the spot rate) and a tradable instrument is *missing* in short rate models.

- Or from the view point of the Harrison-Pliska theorem: Since  $r(t)$  contains less information than  $f(t, T)$  or  $B(t, T)$ , the market is not complete for short rate models.
  - \* Thus the martingale measure in short rate models is *not uniquely determined* by fixing the numeraire instrument.
  - \* Thus, via the Girsanov theorem: **There is freedom in choosing the drift used for pricing!**
- The information lost in the transition s3.6 must be reinserted”by hand”.
  - This is done by directly specifying a drift in the world governed by the *martingale measure*.
  - This is the essential difference in the models here compared to those encountered in the previous chapters where the drift was always specified in the *real* world.
  - **For short rate models the drift is directly chosen for the world governed by the *martingale measure* rather than for the real world!**
- A popular motivation in selecting a drift is the so-called *mean reversion*.

- Interest rates do not rise or fall to arbitrarily high or low levels but tend to oscillate back and forth about a long term mean.
- This can be modeled with a parameter  $v > 0$  in a drift of the functional form

$$\mu - vr$$

- for values of  $r$  small enough so that  $vr < \mu$ , the drift is positive
    - \*  $r$  tends toward larger values.
  - for values of  $r$  large enough so that  $vr > \mu$ , the drift is negative
    - \*  $r$  tends toward smaller values.
  - $r$  thus drifts toward a mean value  $\mu$  at a rate  $\nu$ .
- An example of a *mean reversion model* is the so-called *Hull-White model*[19].
    - This model specifies the following stochastic process *directly* in the *risk-neutral martingale measure*:

$$dr(t) = [\mu(t) - v(t)r] dt + \sigma(t) dW \quad (3.67)$$

- Several "named" models can be distinguished from one another essentially through the form of their drift:
- Normal Models  $dr(t) = a(r, t)dt + b(r, t) dW$ 
  - Stationary Models  $b(r, t) = \sigma$ 
    - \* Arbitrage-free Models
      - Hull-White  $a(r, t) = \mu(t) - vr$  (mean reverting)
      - Ho-Lee  $a(r, t) = \mu(t)$
    - \* Equilibrium Models (not arbitrage because of too few degrees of freedom)
      - Vasiceck  $a(r, t) = \mu - vr$  (mean reverting)
      - Rendleman-Barter  $a(r, t) = \mu$
  - Non-Stationary Models  $b(r, t) = \sigma(t)$
- Lognormal Models  $d \ln r(t) = a(r, t)dt + b(r, t) dW$ 
  - Stationary Models  $b(r, t) = \sigma$

- Non-Stationary Models  $b(r, t) = \sigma(t)$ 
  - \* Arbitrage-free Models
    - Black-Karasinski  $a(r, t) = \mu(t) - v(t) \ln r$  (mean reverting)
    - Black-Derman-Toy  $a(r, t) = \mu(t) - \frac{\partial \sigma(t) / \partial t}{\sigma(t)} \ln r$
- Called stationary or non-stationary depending on whether or not the volatility is a function of time<sup>15</sup>.
- The models allowing for neither a time dependence nor an  $r$  dependence (Vasiceck, Rendleman-Barter) obviously are referred to as *equilibrium models*.
  - They cannot reproduce the current term structure arbitrage-free.
  - Only a "best fit" can be obtained (for example by minimizing the *root mean square error*)
- The volatility of  $dr$  is independent of  $r$  in normal models, and proportional to  $r$  in lognormal models (see 3.54).

---

<sup>15</sup>Ho-Lee and Hull-White are often applied for time-dependent volatilities. Their inventors, however, originally assumed constant volatilities.

- An intermediate scheme is the *Cox-Ingersoll-Ross* model [7] for which the volatility is assumed to be proportional to  $\sqrt{r}$ .

$$dr = (\mu - vr)dt + \sigma\sqrt{r}dW$$

- Note: All of these processes are modelled in reference to the *risk-neutral martingale measure* directly, i.e., should be used directly for pricing *without* first performing a Girsanov-like drift transformation!
- The volatility term is invariant under the Girsanov transformation.
  - Thus the volatility taken for the valuation is the same as that observed in the real world.
- The form taken on by the drift is, however, a result of the particular choice of the measure (coordinate system) for pricing, and as such, a rather artificial construct.
  - It is thus not apparent why a specific form of drift (for example, mean reversion) should be modelled in a specific (for example, risk-neutral) ar-

tificial world (dependent on the selection of a particular numeraire) when our intuitive conception of the drift actually pertains to the *real* world.

- More precisely: According to Girsanov, the process modelled with respect to the martingale measure differs from the real world process by an *arbitrary*<sup>16</sup> previsible process  $\gamma(r, t)$ .

- \* Therefore the drift with respect to a martingale measure provides as good as *no* information about the drift in the *real* world.

- For example, the model 3.67 shows mean reversion in the martingale measure,

- But in the real world it has the form

$$dr(t) = [\mu(t) - v(t)r + \gamma(r, t)] dt + \sigma(t) dW$$

- \* With a (practically) *arbitrary* previsible process  $\gamma$ .

- It is by no means clear why this process should show any mean reversion in the real world.

---

<sup>16</sup>with the restriction that it must satisfy the boundedness condition  $E \left[ \exp \left( \frac{1}{2} \int_0^T \gamma(t) dt \right) \right] < \infty$

- Confused?
  - Just shown the necessity of specifying a drift, i.e. a martingale measure, *explicitly* in short rate models since the martingale measure is not unique.
  - But: In Section 1 we only needed volatility input but *no* drift information!
  - For both the normal and the lognormal models, the interest rate trees could be constructed completely without any drift input.
  
- This stems from the condition 3.27,  $p = 1/2$ , which we introduced "by hand" for the sake of simplicity.
  - Through this choice, we have explicitly selected one particular measure from the family of arbitrage-free martingale measures belonging to the risk-neutral numeraire instrument (the bank account).
  - This completely fixed the drift in accordance with the Girsanov Theorem.
  - One can also recognize this fact explicitly since the drift can be quite simply determined from the generated tree.

- \* For a binomial tree with  $p = 1/2$  that the expectations of the short rate as seen from the node  $(i, j)$  are simply

$$a(i, j)\delta t = \mathbb{E}[r(i, j)] = \frac{1}{2} [r(i+1, j) + r(i, j+1)] \quad \text{normal model}$$

$$a(i, j)\delta t = \mathbb{E}[\ln r(i, j)] = \frac{1}{2} [\ln r(i+1, j) + \ln r(i, j+1)] \quad \text{lognormal model}$$

from which the drift at each node can be immediately determined.

- Had the probability of an up-move not been fixed at  $1/2$ , a free parameter  $p$  would remain unspecified in Equation 3.55.
  - This would allow for many different drift functions.

# 4

## Assignments

1. Prove the product rule in 2.5 by applying Ito's Lemma (in the version for two stochastic variables) to the function  $f(U, Y) = U Y^{-1}$ .
2. Use Ito's Lemma, to derive the process for  $Y^{-1}$  if  $Y$  follows the process 2.3.
3. Show that from

$$d \ln S(t) = \mu dt + \sigma dW$$

it follows via Ito's Lemma

$$dS(t) = \tilde{\mu}S(t) dt + \sigma S(t)dW \quad \text{with } \tilde{\mu} = \mu + \frac{\sigma^2}{2}$$

4. Show that the solution of the stochastic differential equation 2.33 or equivalently of

$$d \ln S(t) = \mu dt + \sigma dW$$

is

$$S(T) = S(t) \exp [\mu(T - t) + \sigma W_{T-t}] \quad \text{with } W_{T-t} \sim N(0, T - t)$$

5. Using 2.37 compute the value of a plain vanilla call with the payoff profile

$$V(S(T), T) = \max \{S(T) - K, 0\} = \max \{e^x S(t) - K, 0\}$$

#### 6. Short Rate Models with Discrete Compounding

The discount factor over a single time period is usually chosen as  $B(t, t + \delta t) = e^{-r(t)\delta t}$  (see, for example Equation 3.31). Intuitively, this means that interest has been paid infinitely often in the reference period  $\delta t$ , and that these payments were then immediately reinvested at the same rate. Strictly speaking, this contradicts

the concept of a tree model, for which time has been *discretized* into intervals of positive length  $\delta t$ , implying by definition that *nothing* can happen in between these times. To be consistent, some people argue that one should rather use *discrete* compounding, allowing the payment and immediate reinvestment of interest solely after each  $\delta t$ . Only in the limiting case  $\delta t \rightarrow 0$  will then the discount factor for continuous compounding be obtained. If we wish to go along this route, we would therefore have to write

$$B(t, t + \delta t) = \frac{1}{1 + r(t)\delta t} \xrightarrow{\delta t \rightarrow 0} e^{-r(t)\delta t}$$

and the discount factor in Equation 3.31 would have the following form:

$$B(i, j) = \frac{1}{1 + r(i, j)\delta t} \tag{4.1}$$

- Normal Models

- Show that the arbitrage condition analogous to Equation 3.58 for the

short rate  $r(0, n)$  at the lowest node in a time slice is

$$B(t, t+(n+1)\delta t) = \sum_{i=0}^n \frac{G(i, n-i)}{1 + r(0, n)\delta t + 2\delta t^{3/2} \sum_{k=1}^i b(i-k, n-i+k-1)}$$

- Determine the rekursion analogous to Equation 3.57 for all other short rates in a time slice.

- Lognormal Models

- Show that the arbitrage condition analogous to Equation 3.63 for the short rate  $r(0, n)$  at the lowest node in a time slice is

$$B(t, t+(n+1)\delta t) = \sum_{i=0}^n \frac{G(i, n-i)}{1 + r(0, n)\delta t \prod_{k=1}^i \alpha(i-k, n-i+k-1)}$$

- Determine the rekursion analogous to Equation 3.62 for all other short rates in a time slice.

7. Write a VBA-Program which builds a short rate tree in a normal model.

- Use some (invented or real) spot term structure as input and assume a constant volatility.

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# Index

- ADP, 127
- Arbitrage Free Market, 8
- Arrow-Debreu Price, 127
- Arrow-Debreu price
  - instantaneous, 129
- backward induction, 107
- Bank Account, 70
- BGM-Modell, 65
- Binomial Tree, 103
- Binomialbaum, 5
- Black-Derman-Toy, 140, 181
- Black-Karasinski, 181
- Brace-Gatarek-Musiela Modell, 65
- Brownian Motion, 25
- Calibration, 166
- Caplet, 85
- Complete Market, 40
- Convexity Adjustment, 88
- Cox-Ingersoll-Ross, 182
- Discount Factor
  - Instantaneous, 103

Equilibrium Model, 181

Factor Model, 64

Floorlet, 85

forward induction, 114

Forward Price, 73

Forward Rate

- instantane, 66
- instantaneous, 65

Forward Rate Agreement, 85

Forward Volatility, 90

Future Price, 73

future styled, 96

Girsanov Theorem, 28

Green's Function, 114

Heath-Jarrow-Morton Modell, 174

Hedge, 14, 21

HJM Modell, 174

Ho-Lee, 139, 180

Hull-White, 179, 180

Importance Sampling, 171

Integral Measure, 3

- Riemann, 3

Interest Rate

- instantaneous, 65

Interest Tree, 124

LIBOR-In-Arrears FRA, 93

LIBOR-In-Arrears Swap, 92

Lognormal Model, 139

Market Price Of Risk, 50

Market Rate Model, 65

Martingale, 4

Martingale Measure, 2

Mean Reversion, 178

Measure, 21

measure  
    forward neutral, 74  
    risk neutral, 70  
Money Market Account, 70  
Money Market Future, 96  
Monte-Carlo on a Tree, 170  
  
Newton-Raphson Method, 153  
Normal Models, 139  
Normalizing, 4  
Numeraire, 5, 20  
  
One-Factor-Model, 25  
option price  
    normalized, 4  
  
Path Dependence  
    Short Rate Models, 118  
Phase Space, 171  
Previsible, 21  
  
previsible, 146  
Previsible Process, 19  
Probability Measure, 3  
  
Relative Price, 21  
Rendleman-Barter, 180  
Risk Neutral  
    Probability, 2  
  
Self Financing, 16, 21  
Short Rate  
    instantaneous, 65  
Short Rate Model, 101  
Spot Rate  
    instantane, 66  
    instantaneous, 65  
  
Term Structure, 63  
    Model, 63  
Tower Law, 33

Tradeable Instrument, 82

Trinomial Tree, 103

Vasiceck, 180

Volatilities

Hull-White, 169

Volatility

Black-Derman-Toy, 169

Ho-Lee, 169

Zero Bond

Instantaneous, 103