Approximation of Continuous Monitoring with Discrete Monitoring Applied to Down—And—Out Options

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1 Introduction

We consider down—and—out options in the Black—Scholes framework. A down—and—out option is a financial contract that guarantees a payment at maturity provided that at the monitoring instances the price of the underlying stock is above the specified barrier. For the case of continuous monitoring the set of monitoring instances is the life of the option, whereas for the case of discrete monitoring the set of monitoring instances is a finite subset of it.

We denote the price of the underlying stock at time t with S_t, where $t \in [0, T]$ and T denotes the maturity of the options. Moreover, we denote the payoff of the options at maturity with $P(S_T)$, and we denote the barrier of the options with B. Finally, we denote the fair values of the options with continuous and discrete monitoring at time $t = 0$ with $V^{(c)}(P, B, T, S_0)$ and $V_n^{(d)}(P, B, T, S_0)$ respectively, where n denotes the number of the discrete equidistant monitoring instances. In their famous paper (1997) the authors Broadie, Glasserman and Kou prove the following continuity correction for down—and—out calls and puts:

$$V_n^{(d)}(P^{\text{Call/Put}}, B, T, S_0) = V^{(c)}(P^{\text{Call/Put}}, B_n, T, S_0) + o\left(\frac{1}{\sqrt{n}}\right). \tag{1.1a}$$

$$B_n := B \exp\left(-\beta\,\sigma\sqrt{\frac{T}{n}}\right). \tag{1.1b}$$

In formula (1.1), β denotes a fixed number ($\beta \approx 0.5826$) and σ denotes the volatility of the underlying stock. In order to give a motivation for the results to be developed in our paper, a few remarks on formula (1.1) are adequate. Firstly, we note that the formula is restricted to calls and puts, and that it has no obvious extension to more general payoffs. Secondly, the residual term $o(1/\sqrt{n})$ gives no information about the quality of the approximation for fixed n and given parameters P, B, T and S_0. It rather gives information about the convergence of the approximation as $n \to \infty$, where the convergence is pointwise w.r.t. P, B, T and S_0. The goal of our paper is to give an estimate for the difference of the fair values of the options with continuous and discrete monitoring that applies to general payoffs P and finite n, where the dependence on the parameters P, B, T and S_0 is explicit. However, we do not give a continuity correction for the barrier, but we rather compare $V^{(c)}(P, B, T, S_0)$ and $V_n^{(d)}(P, B, T, S_0)$ for the same set of parameters P, B, T and S_0. As was pointed out in Broadie et al. (1997), for this case a formula similar to (1.1) holds with residual term $O(1/\sqrt{n})$ instead of $o(1/\sqrt{n})$.

Before we state the desired estimate as a theorem, we first need a few preliminary remarks. We will have to make some restrictive assumptions on the payoff function $P(S)$. A sufficient condition is that $P(S)$ is bounded and has compact support, where $0 \le S < \infty$. This condition is satisfied for puts but not for calls. Therefore, in order to deal with general payoff functions we choose $R > 0$ sufficiently large, and we define:

$$P_R(S) := \begin{cases} P(S) & \text{if } S \le R; \\ 0 & \text{otherwise}. \end{cases} \tag{1.2}$$

In general, given any payoff function $P(S)$, $P_R(S)$ will satisfy the above condition. Next, we exploit the following trivial inequality:

$$\begin{aligned} &|V_n^{(d)}(P, B, T, S_0) - V^{(c)}(P, B, T, S_0)| \\ &\quad \le |V_n^{(d)}(P, B, T, S_0) - V_n^{(d)}(P_R, B, T, S_0)| \\ &\quad + |V_n^{(d)}(P_R, B, T, S_0) - V^{(c)}(P_R, B, T, S_0)| \\ &\quad + |V^{(c)}(P_R, B, T, S_0) - V^{(c)}(P, B, T, S_0)|. \end{aligned} \tag{1.3}$$

We recall that we are working in the Black—Scholes framework. Since the Black—Scholes PDE is a linear and homogeneous diffusion equation and since initial and boundary conditions are homogeneous, too, the following estimates hold:

$$
\begin{aligned}
&|V^{(c)}(P,B,T,S_0) - V^{(c)}(P_R,B,T,S_0)| \\
&\quad \le V^{(c)}(|P-P_R|,B,T,S_0) \\
&\quad \le V^{(c)}(|P-P_R|,0,T,S_0).
\end{aligned} \tag{1.4a}
$$

$$
\begin{aligned}
&|V_n^{(d)}(P,B,T,S_0) - V_n^{(d)}(P_R,B,T,S_0)| \\
&\quad \le V_n^{(d)}(|P-P_R|,B,T,S_0) \\
&\quad \le V_n^{(d)}(|P-P_R|,0,T,S_0) \\
&\quad = V^{(c)}(|P-P_R|,0,T,S_0).
\end{aligned} \tag{1.4b}
$$

Finally, inserting (1.4) into (1.3) we obtain the following estimate:

$$
\begin{aligned}
&|V_n^{(d)}(P,B,T,S_0) - V^{(c)}(P,B,T,S_0)| \\
&\quad \le |V_n^{(d)}(P_R,B,T,S_0) - V^{(c)}(P_R,B,T,S_0)| + 2\,V^{(c)}(|P-P_R|,0,T,S_0).
\end{aligned} \tag{1.5}
$$

We note that the second term on the right hand side in equation (1.5) is independent of n and B. Therefore, in the remainder of our paper we focus our attention on the first term, i.e. we consider $P_R(S)$ instead of $P(S)$, i.e. we assume w.l.o.g. that the payoff function is bounded and has compact support.

With the above remarks in mind, for the mathematical treatise it is more convenient to work with the heat equation instead of the Black—Scholes equation. Both equations are equivalent, and the transition from one to the other can be found in any good textbook on the subject (cf. Deutsch (2004) or Hull (2000)). We denote the normalised price of the underlying stock with x, $x = 0$ being at the barrier. Moreover, we denote the normalised time to maturity with t, $t = 1$ being today and $t = 0$ being maturity. Moreover, we denote the normalised payoff function with $f(x)$. Finally, we denote the normalised fair values of the down—and—out options with continuous and discrete monitoring with $u(x,t)$ and $u^{(n)}(x,t)$ respectively, where n denotes the number of equidistant discrete monitoring instances as before. Our goal is to prove that $u^{(n)}(x,1)$ converges to $u(x,1)$ as $n \to \infty$, where we consider L^p—convergence as well as uniform convergence w.r.t. x.

2 Statement of the Theorem

Let $1 \le p \le \infty$, and let $f \in L^p(\mathbb{R})$ with $f(x) \ge 0$ and $f(x) = 0$ for all $x \le 0$. We assume that u is the solution to the following Dirichlet type PDE problem:

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ((x,t) \in (0,\infty) \times (0,1)). \tag{2.1a}
$$

$$
u(x,0) = f(x) \qquad (x \in (0,\infty)). \tag{2.1b}
$$

$$
u(0,t) = 0 \qquad (t \in (0,1)). \tag{2.1c}
$$

Moreover, we assume that $u^{(n)}$ is the solution to the following Cauchy type PDE problem:

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \left((x,t) \in \mathbb{R} \times \left(\frac{k-1}{n}, \frac{k}{n}\right);\ k = 1,\dots,n\right). \tag{2.2a}
$$

$$
u(x,0) = f(x) \qquad (x \in \mathbb{R}). \tag{2.2b}
$$

$$
u\left(x, \frac{k}{n}\right) = \begin{cases} \lim\limits_{\xi \to 0+} u(x, \frac{k}{n} - \xi) & (x \in (0,\infty)) \\ 0 & (x \in (-\infty, 0]) \end{cases} \qquad (k = 1,\dots,n). \tag{2.2c}
$$

Theorem 1

The following estimate holds $\forall n \in \mathbb{N} \forall \epsilon \ge 0$:

$$
\|u(\cdot,1) - u^{(n)}(\cdot,1)\|_{L^p(0,\infty)} \le (\epsilon + n\,\exp(-\epsilon^2 n))\,\|f\|_{L^p(0,\infty)}. \tag{2.3}
$$

Corollary 2

(a) *Let* $1 \le p < \infty$. *Then the following convergence holds:*

$$
u^{(n)}(\cdot,1) \overset{n\to\infty}{\longrightarrow} u(\cdot,1) \quad \textit{pointwise a.e.}. \tag{2.4}
$$

(b) Let $p = \infty$. *Then the following convergence holds:*

$$
u^{(n)}(\cdot,1) \overset{n\to\infty}{\longrightarrow} u(\cdot,1) \quad \textit{uniformly}. \tag{2.5}
$$

Proof

Statement (a) is evident from theorem 1. Statement (b) follows from theorem 1 and the continuity of $u(\cdot,1)$ and $u^{(n)}(\cdot,1)$.

3 Proof of the Theorem

Let $\epsilon \ge 0$, and let $v^{(\epsilon)}$ be the solution to the following auxiliary PDE problem:

$$
\frac{\partial v^{(\epsilon)}}{\partial t} = \frac{\partial^2 v^{(\epsilon)}}{\partial x^2} \qquad ((x,t) \in (-\epsilon,\infty) \times (0,1)). \tag{3.1a}
$$

$$
v^{(\epsilon)}(x,0) = f(x) \qquad (x \in (-\epsilon,\infty)). \tag{3.1b}
$$

$$
v^{(\epsilon)}(-\epsilon,t) = 0 \qquad (t \in (0,1)). \tag{3.1c}
$$

Moreover, let $v^{(\epsilon,n)}$ be the solution to the following auxiliary PDE problem:

$$
\frac{\partial v^{(\epsilon,n)}}{\partial t} = \frac{\partial^2 v^{(\epsilon,n)}}{\partial^2} \qquad \left((x,t) \in (-\epsilon,\infty) \times \left(\frac{k-1}{n}, \frac{k}{n}\right);\ k = 1,\dots,n\right). \tag{3.2a}
$$

$$
v^{(\epsilon,n)}(x,0) = f(x) \qquad (x \in (-\epsilon,\infty)). \tag{3.2b}
$$

$$
v^{(\epsilon,n)}\left(x, \frac{k}{n}\right) = \begin{cases} \lim\limits_{\xi \to 0+} v^{(\epsilon,n)}(x, \frac{k}{n} - \xi) & (x \in (0,\infty)) \\ 0 & (x \in (-\epsilon, 0]) \end{cases} \qquad (k = 1,\dots,n). \tag{3.2c}
$$

$$
v^{(\epsilon,n)}(-\epsilon,t) = 0 \qquad \left(t \in \left(\frac{k-1}{n}, \frac{k}{n}\right);\ k = 1,\dots,n\right). \tag{3.2d}
$$

With the help of the maximum principle for parabolic equations we obtain the following estimate $\forall\, x \in (0,\infty)$:

$$
0 \le u(x,1) \le v^{(\epsilon,n)}(x,1) \le v^{(\epsilon)}(x,1). \tag{3.3}
$$

This implies the following estimate:

$$\begin{aligned}&\|u(\cdot,1)-u^{(n)}(\cdot,1)\|_{L^p(0,\infty)}\\&\quad\le\|u(\cdot,1)-v^{(\epsilon,n)}(\cdot,1)\|_{L^p(0,\infty)}+\|v^{(\epsilon,n)}(\cdot,1)-u^{(n)}(\cdot,1)\|_{L^p(0,\infty)}\\&\quad\le\|u(\cdot,1)-v^{(\epsilon)}(\cdot,1)\|_{L^p(0,\infty)}+\|v^{(\epsilon,n)}(\cdot,1)-u^{(n)}(\cdot,1)\|_{L^p(0,\infty)}.\end{aligned}\tag{3.4}$$

The following two Lemmas complete the proof of theorem 1.

Lemma 3

The following estimate holds $\forall\epsilon\ge 0$:

$$\|u(\cdot,1)-v^{(\epsilon)}(\cdot,1)_{L^p(0,\infty)}\le\epsilon\,\|f\|_{L^p(0,\infty)}.\tag{3.5}$$

Proof

Let ϕ denote the heat kernel:

$$\phi(x,t)=\frac{1}{\sqrt{4\pi t}}\,\exp\Big(-\frac{x^2}{4t}\Big).\tag{3.6}$$

By the reflection principle, $v^{(\epsilon)}$ admits the following representation $\forall t\in(0,1]$:

$$v^{(\epsilon)}(x,t)=\int_{-\epsilon}^{\infty}(\phi(x-y,t)-\phi(x+y+2\epsilon,t))f(y)\,\mathrm{d}y.\tag{3.7}$$

By assumption, $f(y)\ge 0$ and $f(y)=0$ for all $y\le 0$. Moreover, $u=v^{(0)}$. This yields:

$$\Big|u(x,1)-v^{(\epsilon)}(x,1)\Big|=\int_0^{\infty}(\phi(x+y,1)-\phi(x+y+2\epsilon,1))f(y)\,\mathrm{d}y.\tag{3.8}$$

We use the following notation:

$$c=\int_0^{\infty}(\phi(x,1)-\phi(x+2\epsilon,1))\,\mathrm{d}x.\tag{3.9a}$$

$$\rho(x)=\frac{1}{c}\,(\phi(x,1)-\phi(x+2\epsilon,1)).\tag{3.9b}$$

Now, equation (3.8) reads:

$$\Big|u(x,1)-v^{(\epsilon)}(x,1)\Big|=c\int_0^{\infty}\rho(x+y)f(y)\,\mathrm{d}y.\tag{3.10}$$

We note that ρ is a probability density on $(0,\infty)$. First, let $1\le p<\infty$. With the help of Jensen's inequality we obtain:

$$\begin{aligned}\int_0^{\infty}\left(\int_0^{\infty}\rho(x+y)f(y)\,\mathrm{d}y\right)^p\mathrm{d}x&\le\int_0^{\infty}\int_0^{\infty}\rho(x+y)\,(f(y))^p\,\mathrm{d}y\,\mathrm{d}x\\&=\int_0^{\infty}\left(\int_0^{\infty}\rho(x+y)\,\mathrm{d}x\right)(f(y))^p\,\mathrm{d}y\le\int_0^{\infty}(f(y))^p\,\mathrm{d}y.\end{aligned}\tag{3.11}$$

Now, let $p=\infty$. Then we have:

$$\begin{aligned}&\sup_{x\in(0,\infty)}\int_0^{\infty}\rho(x+y)f(y)\,\mathrm{d}y\le\left(\sup_{x\in(0,\infty)}\int_0^{\infty}\rho(x+y)\,\mathrm{d}y\right)(\operatorname*{ess\,sup}_{y\in(0,\infty)}f(y))\\&\quad\le(\operatorname*{ess\,sup}_{y\in(0,\infty)}f(y))\end{aligned}\tag{3.12}$$

By assumption we have:

$$c=\int_0^{2\epsilon}\phi(x,1)\,\mathrm{d}x\le\epsilon.\tag{3.13}$$

Finally, lemma 3 follows from equations (3.10), (3.11), (3.12) and (3.13).

Lemma 4

The following estimate holds $\forall n\in\mathbb{N}\forall\epsilon\ge 0$:

$$\|v^{(\epsilon,n)}(\cdot,1)-u^{(n)}(\cdot,1)\|_{L^p(0,\infty)}\le n\,\exp(-\epsilon^2\,n)\,\|f\|_{L^p(0,\infty)}.\tag{3.14}$$

Proof

Let ϕ denote the heat kernel defined by (3.6). We define linear operators, $A^{(n)}$ and $B^{(n,\epsilon)}$, acting on $L^p(0,\infty)$:

$$A^{(n)}\,v(x)=\int_0^{\infty}\phi\left(x-y,\frac{1}{n}\right)v(y)\,\mathrm{d}y.\tag{3.15a}$$

$$B^{(n,\epsilon)}\,v(x)=\int_0^{\infty}\phi\left(x+y+2\epsilon,\frac{1}{n}\right)v(y)\,\mathrm{d}y.\tag{3.15b}$$

We consider the operator norms of $A^{(n)}$ and $B^{(n,\epsilon)}$. First we note that $\phi(\cdot,t)$ is a probability density on $\mathbb{R}$ for all $t>0$. A calculation similar to (3.11), (3.12) and (3.13) shows:

$$\|A^{(n)}\|_{\mathcal{L}(L^p(0,\infty))}\le 1.\tag{3.16}$$

By assumption we have $\forall t>0\forall x,y\in(0,\infty)$:

$$\phi(x+y+2\epsilon,t)\le\exp(-\epsilon^2\,n)\,\phi(x+y,t)\le\exp(-\epsilon^2\,n)\,\phi(x-y,t).\tag{3.17}$$

Equations (3.16) and (3.17) imply:

$$\|B^{(n,\epsilon)}\|_{\mathcal{L}(L^p(0,\infty))}\le\exp(-\epsilon^2\,n).\tag{3.18a}$$

$$\|A^{(n)}-B^{(n,\epsilon)}\|_{\mathcal{L}(L^p(0,\infty))}\le 1.\tag{3.18b}$$

With the help of the reflection principle we obtain the following representations:

$$u^{(n)}(x,1)=(A^{(n)})^n\,f(x).\tag{3.19a}$$

$$v^{(n,\epsilon)}(x,1)=(A^{(n)}-B^{(n,\epsilon)})^n\,f(x).\tag{3.19b}$$

This implies:

$$v^{(n,\epsilon)}(x,1) - u^{(n)}(x,1) = -\sum_{k=0}^{n-1} (A^{(n)} - B^{(n,\epsilon)})^k B^{(n,\epsilon)} (A^{(n)})^{n-1-k} f(x). \quad (3.20)$$

Finally, lemma 4 follows from equations (3.16), (3.18) and (3.20).

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