2 Applications of Copulas for the Calculation of Value-at-Risk

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We will focus on the computation of the Value-at-Risk (VaR) from the perspective of the dependency structure between the risk factors. Apart from historical simulation, most VaR methods assume a multivariate normal distribution of the risk factors. Therefore, the dependence structure between different risk factors is defined by the correlation between those factors. It is shown in Embrechts, McNeil and Straumann (1999) that the concept of correlation entails several pitfalls. The authors therefore propose the use of *copulas* to quantify dependence.

For a good overview of copula techniques we refer to Nelsen (1999). Copulas can be used to describe the dependence between two or more random variables with arbitrary marginal distributions. In rough terms, a copula is a function $C : [0,1]^n \rightarrow [0,1]$ with certain special properties. The joint multidimensional cumulative distribution can be written as

$$\begin{aligned} \mathrm{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) &= C\left(\mathrm{P}(X_1 \leq x_1), \ldots, \mathrm{P}(X_n \leq x_n)\right) \\ &= C\left(F_1(x_1), \ldots, F_n(x_n)\right), \end{aligned}$$

where $F_1, \ldots, F_n$ denote the cumulative distribution functions of the n random variables $X_1, \ldots, X_n$. In general, a copula C depends on one or more copula parameters $p_1, \ldots, p_k$ that determine the dependence between the random variables $X_1, \ldots, X_n$. In this sense, the correlation $\rho(X_i, X_j)$ can be seen as a parameter of the so-called Gaussian copula.

Here we demonstrate the process of deriving the VaR of a portfolio using the copula method with XploRe, beginning with the estimation of the selection of the copula itself, estimation of the copula parameters and the computation of the VaR. Backtesting of the results is performed to show the validity and relative quality of the results. We will focus on the case of a portfolio containing two market risk factors only, the FX rates USD/EUR and GBP/EUR. Copulas in more dimensions exist, but the selection of suitable n-dimensional copulas is still quite limited. While the case of two risk factors is still important for applications, e.g. spread trading, it is also the case that can be best described.

As we want to concentrate our attention on the modelling of the dependency structure, rather than on the modelling of the marginal distributions, we restrict our analysis to normal marginal densities. On the basis of our backtesting results, we find that the copula method produces more accurate results than "correlation dependence".

2.1 Copulas

In this section we summarize the basic results without proof that are necessary to understand the concept of copulas. Then, we present the most important properties of copulas that are needed for applications in finance. In doing so, we will follow the notation used in Nelsen (1999).

2.1.1 Definition

DEFINITION 2.1 *A 2-dimensional copula is a function* $C : [0,1]^2 \rightarrow [0,1]$ *with the following properties:*

1. *For every* $u \in [0,1]$
$$C(0,u) = C(u,0) = 0\,. \tag{2.1}$$
2. *For every* $u \in [0,1]$
$$C(u,1) = u \quad \text{and} \quad C(1,u) = u\,. \tag{2.2}$$
3. *For every* $(u_1,u_2),(v_1,v_2) \in [0,1] \times [0,1]$ *with* $u_1 \leq v_1$ *and* $u_2 \leq v_2$*:*
$$C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \geq 0\,. \tag{2.3}$$

A function that fulfills property 1 is also said to be *grounded*. Property 3 is the two-dimensional analogue of a nondecreasing one-dimensional function. A function with this feature is therefore called *2-increasing*.

The usage of the name "copula" for the function C is explained by the following theorem.

2.1.2 Sklar's Theorem

The *distribution function* of a random variable R is a function F that assigns all $r \in \overline{\mathbb{R}}$ a probability $F(r) = \mathrm{P}(R \leq r)$. In addition, the *joint distribution function* of two random variables R_1, R_2 is a function H that assigns all $r_1, r_2 \in \mathbb{R}$ a probability $H(r_1, r_2) = \mathrm{P}(R_1 \leq r_1, R_2 \leq r_2)$.

THEOREM 2.1 (Sklar's theorem) *Let H be a joint distribution function with margins F_1 and F_2. Then there exists a copula C with*

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \tag{2.4}$$

for every $x_1, x_2 \in \overline{\mathbb{R}}$. If F_1 and F_2 are continuous, then C is unique. Otherwise, C is uniquely determined on Range $F_1 \times$ Range F_2. On the other hand, if C is a copula and F_1 and F_2 are distribution functions, then the function H defined by (2.4) is a joint distribution function with margins F_1 and F_2.

It is shown in Nelsen (1999) that H has margins F_1 and F_2 that are given by $F_1(x_1) \stackrel{\text{def}}{=} H(x_1, +\infty)$ and $F_2(x_2) \stackrel{\text{def}}{=} H(+\infty, x_2)$, respectively. Furthermore, F_1 and F_2 themselves are distribution functions. With Sklar's Theorem, the use of the name "copula" becomes obvious. It was chosen by Sklar (1996) to describe "a function that links a multidimensional distribution to its one-dimensional margins" and appeared in mathematical literature for the first time in Sklar (1959).

2.1.3 Examples of Copulas

Product Copula The structure of independence is especially important for applications.

DEFINITION 2.2 *Two random variables R_1 and R_2 are independent if and only if the product of their distribution functions F_1 and F_2 equals their joint distribution function H,*

$$H(r_1, r_2) = F_1(r_1) \cdot F_2(r_2) \quad \textit{for all} \quad r_1, r_2 \in \overline{\mathbb{R}}\,. \tag{2.5}$$

Thus, we obtain the independence copula $C = \Pi$ by

$$\Pi(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i \,,$$

which becomes obvious from the following theorem:

THEOREM 2.2 *Let R_1 and R_2 be random variables with continuous distribution functions F_1 and F_2 and joint distribution function H. Then R_1 and R_2 are independent if and only if $C_{R_1 R_2} = \Pi$.*

From Sklar's Theorem we know that there exists a unique copula C with

$$\mathrm{P}(R_1 \leq r_1, R_2 \leq r_2) = H(r_1, r_2) = C(F_1(r_1), F_2(r_2)) \,. \tag{2.6}$$

Independence can be seen using Equation (2.4) for the joint distribution function H and the definition of Π,

$$H(r_1, r_2) = C(F_1(r_1), F_2(r_2)) = F_1(r_1) \cdot F_2(r_2) \,. \tag{2.7}$$

Gaussian Copula The second important copula that we want to investigate is the *Gaussian* or *normal copula*,

$$C_\rho^{\text{Gauss}}(u, v) \stackrel{\text{def}}{=} \int_{-\infty}^{\Phi_1^{-1}(u)} \int_{-\infty}^{\Phi_2^{-1}(v)} f_\rho(r_1, r_2) dr_2 dr_1 \,, \tag{2.8}$$

see Embrechts, McNeil and Straumann (1999). In (2.8), f_ρ denotes the bivariate normal density function with correlation ρ for $n = 2$. The functions Φ_1, Φ_2 in (2.8) refer to the corresponding one-dimensional, cumulated normal density functions of the margins.

In the case of vanishing correlation, $\rho = 0$, the Gaussian copula becomes

$$\begin{aligned} C_0^{\text{Gauss}}(u, v) &= \int_{-\infty}^{\Phi_1^{-1}(u)} f_1(r_1) dr_1 \int_{-\infty}^{\Phi_2^{-1}(v)} f_2(r_2) dr_2 \\ &= u\, v \\ &= \Pi(u, v) \quad \text{if} \quad \rho = 0 \,. \end{aligned} \tag{2.9}$$

Result (2.9) is a direct consequence of Theorem 2.2.

As $\Phi_1(r_1), \Phi_2(r_2) \in [0, 1]$, one can replace u, v in (2.8) by $\Phi_1(r_1), \Phi_2(r_2)$. If one considers r_1, r_2 in a probabilistic sense, i.e. r_1 and r_2 being values of two random variables R_1 and R_2, one obtains from (2.8)

$$C_\rho^{\text{Gauss}}(\Phi_1(r_1), \Phi_2(r_2)) = \mathrm{P}(R_1 \leq r_1, R_2 \leq r_2) \,. \tag{2.10}$$

In other words: $C_\rho^{\text{Gauss}}(\Phi_1(r_1), \Phi_2(r_2))$ is the binormal cumulated probability function.

Gumbel-Hougaard Copula Next, we consider the *Gumbel-Hougaard* family of copulas, see Hutchinson (1990). A discussion in Nelsen (1999) shows that C_θ is suited to describe bivariate extreme value distributions. It is given by the function

$$C_\theta(u,v) \stackrel{\text{def}}{=} \exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right\}. \tag{2.11}$$

The parameter θ may take all values in the interval $[1,\infty)$.

For $\theta = 1$, expression (2.11) reduces to the product copula, i.e. $C_1(u,v) = \Pi(u,v) = u\,v$. For $\theta \to \infty$ one finds for the Gumbel-Hougaard copula

$$C_\theta(u,v) \stackrel{\theta\to\infty}{\longrightarrow} \min(u,v) \stackrel{\text{def}}{=} M(u,v).$$

It can be shown that M is also a copula. Furthermore, for any given copula C one has $C(u,v) \le M(u,v)$, and M is called the *Fréchet-Hoeffding upper bound*. The two-dimensional function $W(u,v) \stackrel{\text{def}}{=} \max(u+v-1,0)$ defines a copula with $W(u,v) \le C(u,v)$ for any other copula C. W is called the *Fréchet-Hoeffding lower bound*.

2.1.4 Further Important Properties of Copulas

In this section we focus on the properties of copulas. The theorem we will present next establishes the continuity of copulas via a Lipschitz condition on $[0,1]\times[0,1]$:

THEOREM 2.3 *Let C be a copula. Then for every $u_1, u_2, v_1, v_2 \in [0,1]$:*

$$|C(u_2,v_2) - C(u_1,v_1)| \le |u_2 - u_1| + |v_2 - v_1|. \tag{2.12}$$

From (2.12) it follows that every copula C is uniformly continuous on its domain. A further important property of copulas concerns the partial derivatives of a copula with respect to its variables:

THEOREM 2.4 *Let C be a copula. For every $u \in [0,1]$, the partial derivative $\partial C/\partial v$ exists for almost every $v \in [0,1]$. For such u and v one has*

$$0 \le \frac{\partial}{\partial v} C(u,v) \le 1. \tag{2.13}$$

The analogous statement is true for the partial derivative $\partial C/\partial u$.
In addition, the functions $u \to C_v(u) \stackrel{\text{def}}{=} \partial C(u,v)/\partial v$ and $v \to C_u(v) \stackrel{\text{def}}{=} \partial C(u,v)/\partial u$ are defined and nondecreasing almost everywhere on [0,1].

To give an example of this theorem, we consider the partial derivative of the Gumbel-Hougaard copula (2.11) with respect to u,

$$C_{\theta,u}(v) = \frac{\partial}{\partial u} C_\theta(u,v) = \exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right\} \times \\ \left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{-\frac{\theta-1}{\theta}} \frac{(-\ln u)^{\theta-1}}{u}. \quad (2.14)$$

Note that for $u \in (0,1)$ and for all $\theta \in \mathbb{R}$ where $\theta > 1$, $C_{\theta,u}$ is a strictly increasing function of v. Therefore the inverse function $C_{\theta,u}^{-1}$ is well defined. However, as one might guess from (2.14), $C_{\theta,u}^{-1}$ can not be calculated analytically so that some kind of numerical algorithm has to be used for this task. As C_θ is symmetric in u and v, the partial derivative of C_θ with respect to v shows an identical behaviour for the same set of parameters.

We will end this section with a statement on the behaviour of copulas under strictly monotone transformations of random variables.

THEOREM 2.5 *Let R_1 and R_2 be random variables with continuous distribution functions and with copula $C_{R_1 R_2}$. If α_1 and α_2 are strictly increasing functions on Range R_1 and Range R_2, then $C_{\alpha_1(R_1)\,\alpha_2(R_2)} = C_{R_1 R_2}$. In other words: $C_{R_1 R_2}$ is invariant under strictly increasing transformations of R_1 and R_2.*

2.2 Computing Value-at-Risk with Copulas

Now that we have given the most important properties of copulas, we turn to the practical question of how to compute the Value-at-Risk of a portfolio using copulas. The following steps need to be performed:

2.2.1 Selecting the Marginal Distributions

The copula method works with any given marginal distribution, i.e. it does not restrict the choice of margins. However, we will use normal margins for simplicity and in order to allow a comparison with standard VaR methods.

2.2.2 Selecting a Copula

A wide variety of copulas exists, mainly for the two dimensional case (Nelsen (1999)). In our numerical tests, we will use some of the copulas presented in Table 4.1 of Nelsen (1999) in our experiments for comparison which are implemented in the function

`C = VaRcopula(uv,theta,0,copula)`
returns $C_\theta(u, v)$ for copula `copula` with parameter $\theta =$ `theta`. `uv` is a $n \times 2$ vector of coordinates, where the copula is calculated.

For easy reference the implemented copulas are given in Table 2.1.

2.2.3 Estimating the Copula Parameters

After selecting a copula we fit the copula to a time series

$$s = s^{(1)}, \ldots, s^{(T)} \text{ with } s^{(t)} = (s_1^{(t)}, \ldots, s_n^{(t)})$$

for $t \in 1, \ldots, T$. For simplicity we assume that the $s^{(t)}$ are realizations of i.i.d. random variables $S^{(t)}$. The first step will be to determine the parameters of the marginal distributions. In the numerical example we will use the normal distribution $\mathrm{N}(0, \sigma_i^2)$, and estimate the volatility σ_i using an equally weighted volatility estimator $\hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=2}^{T}(r_i^{(t)})^2$ of the returns $r_i^{(t)} = \log(s_i^{(t)}/s_i^{(t-1)})$ for simplicity. The marginal distributions of the risk factors are then log-normal. The remaining task is to estimate the copula parameters. In the XploRe `VaR` quantlib this is done by the function

`res = VaRfitcopula(history,copula,method)`
fits the `copula` to the `history` using fitting function `method`. The result `res` is a list containing the estimates of the copula parameter together with there standard deviations.

Least Square Fit The main idea of the least square fit is that the cumulative distribution function $F_\theta^{(C)}(x)$ defined by the copula C should fit the sample

Table 2.1. Copulas implemented in the `VaR` quantlib.

#	$C_\theta(u,v) =$	$\theta \in$
1	$\max\left([u^{-\theta}+v^{-\theta}-1]^{-1/\theta},0\right)$	$[-1,\infty)\backslash\{0\}$
2	$\max\left(1-[(1-u)^\theta+(1-v)^\theta-1]^{1/\theta},0\right)$	$[1,\infty)$
3	$\frac{uv}{1-\theta(1-u)(1-v)}$	$[-1,1)$
4	$\exp\left(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta}\right)$	$[1,\infty)$
5	$-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$	$(-\infty,\infty)\backslash\{0\}$
6	$1-\left[(1-u)^\theta+(1-v)^\theta-(1-u)^\theta(1-v)^\theta\right]^{1/\theta}$	$[1,\infty)$
7	$\max\left[\theta uv+(1-\theta)(u+v-1),0\right]$	$(0,1]$
8	$\max\left[\frac{\theta^2uv-(1-u)(1-v)}{\theta^2-(\theta-1)^2(1-u)(1-v)},0\right]$	$(0,1]$
9	$uv\exp(-\theta\ln u\ln v)$	$(0,1]$
10	$uv/\left[1+(1-u^\theta)(1-v^\theta)\right]^{1/\theta}$	$(0,1]$
11	$\max\left(\left[u^\theta v^\theta-2(1-u^\theta)(1-v^\theta)\right]^{1/\theta},0\right)$	$(0,1/2]$
12	$\left(1+\left[(u^{-1}-1)^\theta+(v^{-1}-1)^\theta\right]^{1/\theta}\right)^{-1}$	$[1,\infty)$
13	$\exp\left(1-\left[(1-\ln u)^\theta+(1-\ln v)^\theta-1\right]^{1/\theta}\right)$	$(0,\infty)$
14	$\left(1+\left[(u^{-1/\theta}-1)^\theta+(v^{-1/\theta}-1)^\theta\right]^{1/\theta}\right)^{-\theta}$	$[1,\infty)$
15	$\max\left(\left\{1-\left[(1-u^{1/\theta})^\theta+(1-v^{1/\theta})^\theta\right]^{1/\theta}\right\}^\theta,0\right)$	$[1,\infty)$
16	$\frac{1}{2}\left(S+\sqrt{S^2+4\theta}\right)$	$[0,\infty)$
	$\hookrightarrow S=u+v-1-\theta\left(\frac{1}{u}+\frac{1}{v}-1\right)$	
21	$1-\left(1-\left\{\max(S(u)+S(v)-1,0)\right\}^\theta\right)^{\frac{1}{\theta}}$	$[1,\infty)$
	$\hookrightarrow S(u)=\left[1-(1-u)^\theta\right]^{1/\theta}$	

distribution function $S(x) = \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}(s_1^{(t)} \le x_1, \ldots, s_n^{(t)} \le x_n)$ as close as possible in the mean square sense. The function $\mathbf{1}(A)$ is the indicator function of the event A. In order to solve the least square problem on a computer, a discretization of the support of $F_\theta^{(C)}$ is needed, for which the sample set $s^{(t)}$ seems to be well suited. The copula parameter estimators are therefore the solution of the following minimization problem:

$$\min \sum_{t=1}^{T} \left(F_\theta^{(c)}(s^{(t)}) - S(s^{(t)}) + \frac{1}{2T} \right)^2 \text{ subject to } \theta \in D_C \, .$$

using the Newton method on the first derivative ($\texttt{method} = 1$). The addition of $\frac{1}{2T}$ avoids problems that result from the $\frac{1}{T}$ jumps at the sample points. While this method is inherently numerically stable, it will produce unsatisfactory results when applied to risk management problems, because the minimization will fit the copula best where there are the most datapoints, and not necessarily at the extreme ends of the distribution. While this can be somewhat rectified by weighting schemes, the maximum likelihood method does this directly.

Maximum Likelihood The likelihood function of a probability density function $f_\theta^{(C)}(x)$ evaluated for a time series s is given by $l(\theta) = \prod_{t=1}^{T} f_\theta^{(C)}(s^t)$. The maximum likelihood method states that the copula parameters at which l reaches its maximum are good estimators of the “real” copula parameters. Instead of the likelihood function, it is customary to maximize the log-likelihood function

$$\max \sum_{t=1}^{T} \log \left(f_\theta^{(C)}(x^{(t)}) \right) \text{ s.t. } \theta \in D_C \, .$$

Maximization can be performed on the copula function itself by the Newton method on the first derivative ($\texttt{method}$=2) or by an interval search ($\texttt{method}$=3). The true maximum likelihood method is implemented in $\texttt{method}$=4 using an interval search. Depending on the given copula it may not be possible to maximize the likelihood function (i.e. if $f_\theta^{(C)}(s^{(t)})) = 0$ for some t and all θ. In this case the least square fit may be used as a fallback.

2.2.4 Generating Scenarios - Monte Carlo Value-at-Risk

Assume now that the copula C has been selected. For risk management purposes, we are interested in the Value-at-Risk of a position. While analytical methods for the computation of the Value-at-Risk exist for the multivariate normal distribution (i.e. for the Gaussian copula), we will in general have to use numerical simulations for the computation of the VaR. To that end, we need to generate pairs of random variables $(X_1, X_2) \sim F^{(C)}$, which form scenarios of possible changes of the risk factor. The Monte Carlo method generates a number N of such scenarios, and evaluates the present value change of a portfolio under each scenario. The sample $\alpha-$quantile is then the one period Value-at-Risk with confidence α.

Our first task is to generate pairs (u, v) of observations of $U(0, 1)$ distributed random variables U and V whose joint distribution function is $C(u, v)$. To reach this goal we use the method of conditional distributions. Let c_u denote the conditional distribution function for the random variable V at a given value u of U,

$$c_u(v) \stackrel{\text{def}}{=} \mathrm{P}(V \le v, U = u) \ . \tag{2.15}$$

From (2.6) we have

$$c_u(v) = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial}{\partial u} C(u, v) = C_u(v) \ , \tag{2.16}$$

where C_u is the partial derivative of the copula. From Theorem 2.4 we know that $c_u(v)$ is nondecreasing and exists for almost all $v \in [0, 1]$.

For the sake of simplicity, we assume from now on that c_u is strictly increasing and exists for all $v \in [0, 1]$. If these conditions are not fulfilled, one has to replace the term “inverse” in the remaining part of this section by “quasi-inverse”, see Nelsen (1999).

With result (2.16) at hand we can now use the method of variable transformation to generate the desired pair (u, v) of pseudo random numbers (PRN). The algorithm consists of the following two steps:

- Generate two independent uniform PRNs $u, w \in [0, 1]$. u is already the first number we are looking for.
- Compute the inverse function of c_u. In general, it will depend on the parameters of the copula and on u, which can be seen, in this context, as an additional parameter of c_u. Set $v = c_u^{-1}(w)$ to obtain the second PRN.

It may happen that the inverse function cannot be calculated analytically. In this case one has to use a numerical algorithm to determine v. This situation occurs for example when Gumbel-Hougaard copulas are used.

v = VaRcopula(uv,theta,-1,copula)
returns inverse $v = c_u^{-1}$ such that $res = c_u(u,v)$ for copula copula with parameter $\theta =$ theta. uv is a $n \times 2$ vector of coordinates, where the copula is calculated.

Finally we determine $x_1 = \Phi_1^{-1}(u)$ and $x_2 = \Phi_2^{-1}(v)$ to obtain one pair (x_1, x_2) of random variables with the desired copula dependence structure. For a Monte Carlo simulation, this procedure is performed N times to yield a sample $X = (x^{(1)}, \dots, x^{(N)})$.

X = VaRsimcopula(N, sigma_1, sigma_2, theta, copula)
returns a sample of size N for the copula copula with parameter $\theta =$ theta and normal distributions with standard deviations $\sigma_1 =$ sigma_1, $\sigma_2 =$ sigma_2.

If we assume a linear position a with holdings $a_1, \dots, a_n$ in each of the risk factors, the change in portfolio value is approximately $\sum_{i=1}^{n} a_i \cdot x_i$. Using a first order approximation, this yields a sample Value-at-Risk with confidence level α.

VaR = VaRestMCcopula(history,a,copula,opt)
fits the copula copula to the history history and returns the N-sample Monte Carlo Value-at-Risk with confidence level $\alpha =$ alpha for position a. N and alpha are contained in list opt.

2.3 Examples

In this section we show possible applications for the Gumbel-Hougaard copula, i.e. for copula = 4. First we try to visualize $C_4(u, v)$ in Figure 2.1.

XFGaccvar1.xpl

In the next Figure 2.2 we show an example of copula sampling for fixed pa-

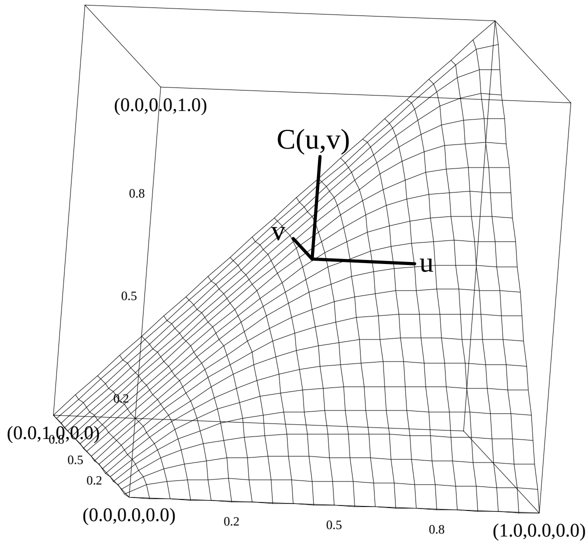

Figure 2.1. Plot of $C_4(u, v)$ for $\theta = 3$

rameters $\sigma_1 = 1$, $\sigma_2 = 1$, $\theta = 3$ for copulas numbered 4, 5, 6, and 12, see Table 2.1.

`XFGaccvar2.xpl`

In order to investigate the connection between the Gaussian and Copula based dependency structure we plot θ against correlation ρ in Figure 2.3. We assume that `tmin` and `tmax` hold the minimum respectively maximum possible θ values. Those can also be obtained by `tmin=VaRcopula(0,0,0,8,copula)` and `tmax=VaRcopula(0,0,0,9,copula)`. Care has to be taken that the values are finite, so we have set the maximum absolute θ bound to 10.

`XFGaccvar3.xpl`

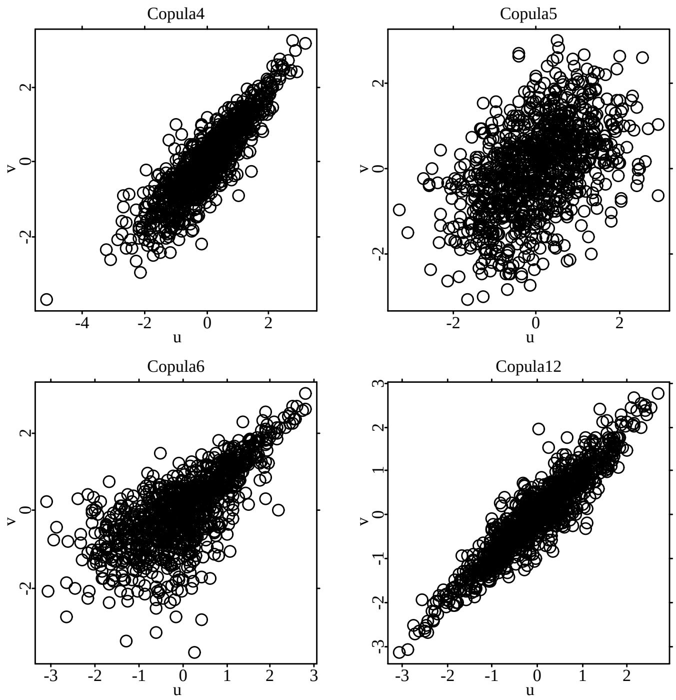

Figure 2.2. 10000-sample output for $\sigma_1 = 1$, $\sigma_2 = 1$, $\theta = 3$

2.4 Results

To judge the effectiveness of a Value-at-Risk model, it is common to use backtesting. A simple approach is to compare the predicted and empirical number of outliers, where the actual loss exceeds the VaR. We implement this test in a two risk factor model using real life time series, the FX rates USD/EUR and GBP/EUR, respectively their DEM counterparts before the introduction of the Euro. Our backtesting investigation is based on a time series ranging from 2 Jan. 1991 until 9 Mar. 2000 and simple linear portfolios $i = 1, \ldots, 4$:

$$\text{Value}(a_i, t)[EUR] = a_{i,1} \times \text{USD}_t - a_{i,2} \times \text{GBP}_t \ . \tag{2.17}$$

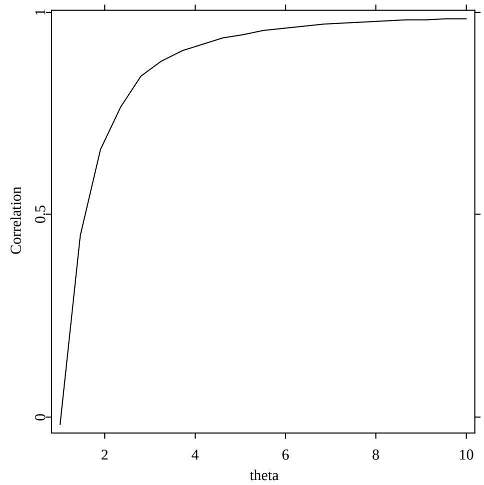

Figure 2.3. Plot of θ against correlation ρ for C_4.

The Value-at-Risk is computed with confidence level $1-\alpha_i$ ($\alpha_1 = 0.1$, $\alpha_2 = 0.05$, and $\alpha_3 = 0.01$) based on a time series for the statistical estimators of length $T = 250$ business days. The actual next day value change of the portfolio is compared to the VaR estimate. If the portfolio loss exceeds the VaR estimate, an outlier has occurred. This procedure is repeated for each day in the time series.

The prediction error as the absolute difference of the relative number of outliers $\hat{\alpha}$ to the predicted number α is averaged over different portfolios and confidence levels. The average over the portfolios ($a_1 = (-3, -2)$ $a_2 = (+3, -2)$ $a_3 = (-3, +2)$ $a_4 = (+3, +2)$) uses equal weights, while the average over the confidence levels i emphasizes the tails by a weighting scheme w_i ($w_1 = 1$, $w_2 = 5$, $w_3 = 10$). Based on the result, an overall error and a relative ranking of the different methods is obtained (see Table 2.2).

As benchmark methods for Value-at-Risk we use the variance-covariance (vcv) method and historical simulation (his), for details see Deutsch and Eller (1999). The variance covariance method is an analytical method which uses a multivariate normal distribution. The historical simulation method not only includes the empirical copula, but also empirical marginal distributions. For the copula VaR methods, the margins are assumed to be normal, the only difference between the copula VaR's is due to different dependence structures (see Table 2.1). Mainly as a consequence of non-normal margins, the historical simulation has the best backtest results. However, even assuming normal margins, certain copulas (5, 12–14) give better backtest results than the traditional variance-covariance method.

				Copula as in Table 2.1																
$\alpha=$	$a=$	his	vcv	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	21
.10	a_1	.103	.084	.111	.074	.100	.086	.080	.086	.129	.101	.128	.129	.249	.090	.087	.084	.073	.104	.080
.05	a_1	.053	.045	.066	.037	.059	.041	.044	.040	.079	.062	.076	.079	.171	.052	.051	.046	.038	.061	.041
.01	a_1	.015	.019	.027	.013	.027	.017	.020	.016	.032	.027	.033	.034	.075	.020	.022	.018	.015	.027	.018
.10	a_2	.092	.078	.066	.064	.057	.076	.086	.062	.031	.049	.031	.031	.011	.086	.080	.092	.085	.065	.070
.05	a_2	.052	.044	.045	.023	.033	.041	.049	.031	.012	.024	.012	.013	.003	.051	.046	.054	.049	.039	.032
.01	a_2	.010	.011	.016	.002	.007	.008	.009	.006	.002	.002	.002	.002	.001	.015	.010	.018	.025	.011	.005
.10	a_3	.099	.086	.126	.086	.064	.088	.096	.073	.032	.054	.033	.031	.016	.094	.086	.105	.133	.070	.086
.05	a_3	.045	.048	.093	.047	.032	.052	.050	.040	.017	.026	.017	.016	.009	.049	.047	.058	.101	.034	.050
.01	a_3	.009	.018	.069	.018	.012	.018	.016	.012	.007	.009	.006	.006	.002	.018	.015	.018	.073	.013	.020
.10	a_4	.103	.090	.174	.147	.094	.095	.086	.103	.127	.094	.129	.127	.257	.085	.085	.085	.136	.088	.111
.05	a_4	.052	.058	.139	.131	.056	.060	.058	.071	.084	.068	.084	.085	.228	.053	.054	.051	.114	.053	.098
.01	a_4	.011	.020	.098	.108	.017	.025	.025	.035	.042	.056	.041	.042	.176	.016	.017	.016	.087	.015	.071
.10	Avg	.014	.062	.145	.123	.085	.055	.052	.082	.193	.104	.194	.194	.478	.045	.061	.045	.110	.082	.075
.05	Avg	.011	.021	.154	.124	.051	.030	.016	.060	.134	.080	.132	.136	.387	.006	.012	.017	.127	.041	.075
.01	Avg	.007	.029	.169	.117	.028	.031	.032	.036	.065	.071	.065	.067	.249	.029	.025	.029	.160	.026	.083
Avg	Avg	.009	.028	.163	.120	.039	.032	.028	.047	.095	.076	.094	.096	.306	.022	.023	.026	.147	.034	.080
Rank		1	6	18	16	9	7	5	10	14	11	13	15	19	2	3	4	17	8	12

Table 2.2. Relative number of backtest outliers $\hat{\alpha}$ for the VaR with confidence $1-\alpha$, weighted average error $|\hat{\alpha}-\alpha|$ and error ranking. XFGaccvar4.xpl

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